A $W^n_p$-theory of parabolic equations with unbounded leading coefficients on non-smooth domains

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ allowing Hardy inequality

$$\int_{\Omega} |\rho^{-1} g|^2 \, dx \leq C \int_{\Omega} |Dg|^2 \, dx, \quad \forall g \in C_0^\infty(\Omega),$$

where $\rho(x) = \text{dist}(x, \partial \Omega)$. It is a classical result that Hardy inequality holds on Lipschitz domains [18]. There have been many other works concerning Hardy inequality. See, for instance, [2,20] and references therein. We only mention that inequality (1.1) holds, for instance, $\Omega$ has plump complement, that is, there exist $b, \sigma \in (0, 1]$ such that for any $s \in (0, \sigma]$ and $x \in \partial \Omega$ there exists a point $y \in B_s(x) \cap \Omega^c$ with $\text{dist}(x, \partial \Omega) \geq bs$.

Fix a constant $\alpha \in [0, \infty)$. In this article we study the equation

$$u_t = a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f$$

with the following conditions: $\exists b_0, K > 0$ such that

$$\delta_0 \rho^{-2\alpha}(x) I \leq (a^{ij}(t, x)) \leq K \rho^{-2\alpha}(x) I,$$

$$\sup_{t} |b^i| = o(\rho^{-1-2\alpha}), \quad \sup_{t} |c| = o(\rho^{-2-2\alpha}),$$

where $I$ is the $d \times d$ identity matrix and by (1.4) we mean

$$\begin{align*}
\delta_0 \rho^{-2\alpha}(x)(t, x) & \leq (a^{ij}(t, x)) \leq K \rho^{-2\alpha}(x) I, \\
\sup_{t} |b^i| & = o(\rho^{-1-2\alpha}), \quad \sup_{t} |c| = o(\rho^{-2-2\alpha}).
\end{align*}$$

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Condition (1.3) can be used, for instance, to describe the heat conduction in a rod when the diffusivity of the rod increases (or decreases) dramatically fast near the boundary. Note that the case \( \alpha = 0 \), when the equation is uniformly parabolic and has bounded leading coefficients, is also under our consideration. Obviously (1.4) is satisfied if \( b' \) and \( c \) are bounded, and furthermore it allows those coefficients to blow up near the boundary at a certain rate.

The equation when \( \alpha \in (-1/2, 1/2) \) and \( \partial \Omega \in C^{2,\alpha} \) was studied long time ago in [19], where interior Schauder estimates were obtained on the basis of the study of a Green function. We also refer to [5] for the case when \( \alpha \in (-\infty, 0) \) and \( \partial \Omega \in C^1 \). Our approach is different from that of [19] and is based on Sobolev spaces with weights introduced in [10] and [15]. As in [5], scaling argument and integration by parts are among our main tools. However unlike in [5], since \( \partial \Omega \) is not supposed to be regular enough we do not use the argument of flattening the boundary. Also due to the unboundedness of the coefficients, several arguments used in [5] break down in our case. In particular, we prove the existence of solutions using different ideas.

Needless to say, Eq. (1.2) with various other assumptions of the coefficients has been extensively studied since long ago. We do not want to try to collect all relevant references. We only mention recent articles [1,3,16,17] dealing with the equation with unbounded leading coefficients. In these articles it is assumed that \( \Omega = \mathbb{R}^d \) or \( \partial \Omega \in C^{2,\alpha} \), and the leading coefficients are locally bounded, that is, they are bounded on \( \Omega \cap B_r(x) \) for any \( r > 0 \) and \( x \in \Omega \). Note that (1.3) has nothing to do with such local bound of the coefficients. We are assuming that the leading coefficients blow up near the boundary. Furthermore we allow \( \Omega \) to be a non-Lipschitz domain, and we estimate not only the gradients of solutions but also all the derivatives of any order. In particular the number of derivatives of the solutions can be fractional. Consequently our results are new even when \( \alpha = 0 \).

As usual, \( \mathbb{R}^d \) stands for the Euclidean space of points \( x = (x^1, \ldots, x^d) \), \( \mathbb{R}^d_+ = \{ x \in \mathbb{R}^d : x^1 > 0 \} \) and \( B_r(x) := \{ y \in \mathbb{R}^d : |x - y| < r \} \). For \( i = 1, \ldots, d \), multi-indices \( \beta = (\beta_1, \ldots, \beta_d) \), \( \beta_i \in \{0, 1, 2, \ldots\} \), and functions \( u(x) \) we set

\[
 u_{ij} = \partial u/\partial x^j = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdots D_d^{\beta_d} u, \quad |\beta| = \beta_1 + \cdots + \beta_d.
\]

We also use the notation \( D^m \) for a partial derivative of order \( m \) with respect to \( x \). If we write \( C = C(\ldots) \), this means that the constant \( C \) depends only on what are in parenthesis.

2. Main results

Fix a bounded infinitely differentiable function \( \psi \) defined in \( \Omega \) such that (see, for instance, Lemma 4.13 in [12])

\[
\rho(x) \leq C \psi(x) \leq C \rho(x), \quad \rho^m |D^m \psi| \leq C(m) < \infty. \tag{2.1}
\]

By modifying the coefficients in (1.2), we rewrite Eq. (1.2) in the following form:

\[
 u_t = \psi^{-2u} a^{ij} u_{ij} + \psi^{-2u-1} b^i u_i + \psi^{-2u-2} cu + f. \tag{2.2}
\]

Here \( i \) and \( j \) go from 1 to \( d \), and the coefficients \( a^{ij}, b^i, c \) are Borel measurable functions of \( t, x \).

To describe the assumptions of \( f \) we use Sobolev spaces introduced in [8,10,15]. Let \( p \in (1, \infty) \), \( \gamma \in \mathbb{R} \) and \( H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = (1 - \Delta)^{-\gamma/2} L_p \) be the set of all distributions \( u \) such that \((1 - \Delta)^{\gamma/2} u \in L_p \). Define

\[
 \|u\|_{H^\gamma_p} = \left\| (1 - \Delta)^{\gamma/2} u \right\|_{L_p}.
\]

It is well known that if \( \gamma \) is a nonnegative integer then

\[
 H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = \{ u : Du, \ldots, D^\gamma u \in L_p \}.
\]

Let \( \xi \in C_0^\infty(\mathbb{R}^d) \) be a nonnegative function satisfying

\[
 \sum_{n=-\infty}^{\infty} \xi(e^{n+1}) > c > 0, \quad \forall t \in \mathbb{R}. \tag{2.3}
\]

For \( \in \Omega \) and \( n \in \mathbb{Z} := \{0, \pm 1, \ldots\} \) define

\[
 \zeta_n(x) = \xi(e^n \psi(x)).
\]

Then \( \text{supp} \; \zeta_n \subset \{ x \in \Omega : e^{-n-k} < \rho(x) < e^{-n+k} \} \) for some \( k > 0 \),

\[
 \sum_{n=-\infty}^{\infty} \zeta_n(x) > c > 0, \quad \zeta_n \in C_0^\infty(\Omega), \quad |D^m \zeta_n(x)| \leq N(\xi) e^{mn}. \tag{2.4}
\]

By \( H^\gamma_{p,\delta}(\Omega) \) we denote the set of all distributions \( u \) on \( \Omega \) such that
We remark that the space $H^\infty_p(\Omega)$ is independent of the choice of $\zeta$ and $\psi$. In particular, if $\gamma = n$ is a nonnegative integer then

$$ L_{p,\theta}(\Omega) := H^0_{p,\theta}(\Omega) = L_p(\Omega, \rho^{\theta - d} dx), $$

$$ H^\gamma_{p,\theta}(\Omega) := \{ u: \ u, \rho Du, \ldots, \rho^n D^n u \in L_{p,\theta}(\Omega) \}, $$

$$ \| u \|_{H^\gamma_{p,\theta}(\Omega)} := \sum_{|\alpha| \leq n} \| \rho^{|\alpha|} D^\alpha u \|_{L_p(\Omega, \rho^{\theta - d} dx)}. \tag{2.6} $$

We remark that the space $H^\infty_{p,\theta}(\Omega)$ is different from $W^{n,p}(\Omega, \rho, \varepsilon)$ introduced in [12], where

$$ W^{n,p}(\Omega, \rho, \varepsilon) = \{ u: \ u, Du, \ldots, D^n u \in L_p(\Omega, \rho^\varepsilon dx) \}. $$

Here are some properties of the space $H^\gamma_{p,\theta}(\Omega)$ taken from [15] (also see [9,10]).

**Lemma 2.1.**

(i) The space $C^\infty_0(\Omega)$ is dense in $H^\gamma_p(\Omega)$.

(ii) Assume that $\gamma - d/p = m + \nu$ for some $m = 0, 1, \ldots$ and $\nu \in (0, 1]$, and $i, j$ are multi-indices such that $|i| \leq m, |j| = m$. Then for any $u \in H^\gamma_{p,\theta}(\Omega)$, we have

$$ \psi^{i+\theta/p} D^i u \in C(\Omega), \quad \psi^{m+\theta/p} D^i u \in C^\nu(\Omega), $$

$$ |\psi^{i+\theta/p} D^i u|_{C(\Omega)} + |\psi^{m+\theta/p} D^i u|_{C^\nu(\Omega)} \lesssim \| u \|_{H^\gamma_{p,\theta}(\Omega)}. \tag{2.6} $$

(iii) $\psi D, D \psi : H^\gamma_{p,\theta}(\Omega) \to H^{-\gamma - 1}_{p,\theta}(\Omega)$ are bounded linear operators, and for any $u \in H^\gamma_{p,\theta}(\Omega)$

$$ \| u \|_{H^\gamma_{p,\theta}(\Omega)} \leq C \| \psi u \|_{H^{-\gamma - 1}_{p,\theta}(\Omega)} + C \| u \|_{H^\gamma_{p,\theta}(\Omega)} \leq C \| u \|_{H^\gamma_{p,\theta}(\Omega)}, $$

$$ \| u \|_{H^\gamma_{p,\theta}(\Omega)} \leq C \| (\psi u) \|_{H^{-\gamma - 1}_{p,\theta}(\Omega)} + C \| u \|_{H^\gamma_{p,\theta}(\Omega)} \leq C \| u \|_{H^\gamma_{p,\theta}(\Omega)} \leq N \| u \|_{H^\gamma_{p,\theta}(\Omega)}. $$

(iv) For any $\nu, \gamma \in \mathbb{R}, \psi^{-\nu} H^\gamma_{p,\theta}(\Omega) = H^\gamma_{p,\theta-\nu}(\Omega)$ and

$$ \| u \|_{H^\gamma_{p,\theta-\nu}(\Omega)} \leq C \| \psi^{-\nu} u \|_{H^\gamma_{p,\theta}(\Omega)} \leq C \| u \|_{H^\gamma_{p,\theta}(\Omega)}. $$

(v) If $\gamma \in (\gamma_0, \gamma_1)$ and $\theta \in (\theta_0, \theta_1)$, then

$$ \| u \|_{H^\gamma_{p,\theta}(\Omega)} \leq \varepsilon \| u \|_{H^\gamma_{p,\theta}(\Omega)} + C(\gamma, p, \varepsilon) \| u \|_{H^0_{p,\theta}(\Omega)}, $$

$$ \| u \|_{H^\gamma_{p,\theta}(\Omega)} \leq C \| u \|_{H^\gamma_{p,\theta}(\Omega)} + C(\gamma, p, \varepsilon) \| u \|_{H^\gamma_{p,\theta}(\Omega)}. $$

Denote

$$ \mathcal{H}^\gamma_p(T) = L_p((0, T], H^\gamma_p), \quad \mathcal{H}^\gamma_{p,\theta}(\Omega, T) = L_p((0, T], H^\gamma_{p,\theta}(\Omega)). $$

$$ L_{-\ldots} = \mathcal{H}^0_{-\ldots}, \quad U^\gamma_p = H^\gamma_{p-2/p}, $$

$$ U^\gamma_{p,\theta,\alpha}(\Omega) = \psi^{\frac{1}{2}(\alpha-1)+1} H^\gamma_{p,\theta}(\Omega) = H^\gamma_{p,\theta+2(\alpha+1)-p}(\Omega). $$

By Lemma 2.1(iii), the operators

$$ \psi^{-2\alpha} \Delta, \ \psi^{-2\alpha - 1} D: \psi^{\mathcal{H}^\gamma+2}_{p,\theta,\alpha}(\Omega, T) \to \psi^{-1-2\alpha} \mathcal{H}^\gamma_{p,\theta}(\Omega, T) $$

are bounded. Thus we naturally introduce the space of solutions of Eq. (2.2) as follows. We write $u \in \mathcal{S}^\gamma_{p,\theta,\alpha}(\Omega, T)$ if $u \in \psi^{\mathcal{H}^\gamma+2}_{p,\theta,\alpha}(\Omega, T), u(0, \cdot) \in U^\gamma_{p,\theta,\alpha}(\Omega)$ and for some $f \in \psi^{-1-2\alpha} \mathcal{H}^\gamma_{p,\theta}(\Omega, T)$

$$ u_t = f $$

in the sense of distributions. In other words, for any $\phi \in C^\infty_0(\Omega)$, the equality
holds for all $t \leq T$. Define

$$\|u\|_{H^{\alpha+2}_{p,0,a}(\Omega,T)} = \|\psi^{-1} u\|_{H^{\alpha+2}_{p,0}(\Omega,T)} + \|\psi^{1 + 2\alpha} u_t\|_{L^p_{p,0}(\Omega,T)} + \|u(0,\cdot)\|_{H^{\alpha+2}_{p,0,a}(\Omega)}.$$ 

**Lemma 2.2.**

(i) Let $2/p < \beta \leq 1$ and $u \in H^{\alpha}_{p,0,a}(\Omega,T)$, then

$$\left[\psi^{\beta(1+\alpha)-1} u\right]_{C^0([0,T],H^{\beta-\alpha}_{p,0} (\Omega))} \leq C \|u\|_{H^{\alpha}_{p,0,a}(\Omega,T)},$$

where $C$ is independent of $T$ and $u$.

(ii) Let $p \in [2, \infty)$, then

$$\sup_{t \leq T} \|\psi^\alpha u(t)\|_{H^{-1}_{p,0} (\Omega)} \leq C \|u\|_{H^{\alpha}_{p,0,a}(\Omega,T)},$$

where $C = C(d, p, \gamma, \theta, T)$. In particular, for any $t \leq T$,

$$\|\psi^\alpha u\|_{L^p_{p,0} (\Omega,T)} \leq C \int_0^t \|u\|_{H^{\alpha}_{p,0,a}(\Omega,s)} ds.$$ 

**Proof.** Let $u_t = f$, $u(0) = u_0$ and $v = \beta/2 - 1/p$. By (2.5) and Lemma 2.1(iv),

$$I := \left[\psi^{\beta(1+\alpha)-1} u\right]_{C^0([0,T],H^{\beta-\alpha}_{p,0} (\Omega))} \leq C \sum_n e^{n(0+p(\beta+\beta\alpha-1))} \|u(t, e^n x) \psi_n e^n x\|_{C^0([0,T],H^{\beta-\alpha}_{p,0} (\Omega))}.$$ 

By Corollary 7.5 in [7], there exists a constant $C > 0$, independent of $T$ and $u$, so that for any $a > 0$,

$$\left[\psi^\alpha u(t)\right]_{H^{-1}_{p,0} (\Omega)} \leq C a^{\beta-1} (\|u(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)} + a^{-1} \|f(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)}).$$

Take $a$ so that $e^\beta = e^{-n(\beta+\beta\alpha)}$, then (2.8) yields

$$I \leq C \sum_n e^{n(0-p)} \|u(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)} + C \sum_n e^{n(0+p(1+2\alpha))} \|f(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)}.$$

Thus (i) is proved. Also if $p > 2$, (ii) follows from (i). But for the case $p = 2$, we prove differently. Obviously

$$\sup_{t \leq T} \|\psi^\alpha u(t)\|_{H^{-1}_{p,0} (\Omega)} \leq C \sum_n e^{n(0+p\alpha)} \sup_{t \leq T} \|u(t, e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)}.$$ 

By Remark 4.14 in [7], for any $a > 0$,

$$\sup_{t \leq T} \|u(t, e^n x) \psi_n e^n x\|_{H^{-1}_{p,0} (\Omega)} \leq C (\|u(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)} + a^{-1} \|f(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)} + \|u_0(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)}).$$

Take $a = e^{-n(1+\alpha)}$, then

$$\sup_{t \leq T} \|\psi^\alpha u(t)\|_{H^{-1}_{p,0} (\Omega)} \leq C \sum_n e^{n(0-p)} \|u(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)} + C \sum_n e^{n(0+p(1+2\alpha))} \|f(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)} + C \sum_n e^{n(0+p\alpha)} \|u_0(e^n x) \psi_n e^n x\|_{H^{\alpha}_{p,0,a}(\Omega,T)}.$$ 

This, with Lemma 2.1(iv) and Remark 3.2, proves the lemma. 

We restate conditions (1.3) and (1.4) as follows (remember that we are considering Eq. (2.2) instead of Eq. (1.2)).
Assumption 2.3.

(i) For any \( \lambda \in \mathbb{R}^d \),
\[
\delta_0|\lambda|^2 \leq a^{ij}(t, x)\lambda^i\lambda^j \leq K|\lambda|^2.
\]
(2.9)

(ii) There is a control on the behavior of \( b^i, c \) near \( \partial \Omega \). Namely,
\[
\lim_{\rho(x) \rightarrow 0} \sup_{t, x \in \Omega} (|b^i(t, x)| + |c(t, x)|) = 0.
\]
(2.10)

Denote \( \rho(x, y) = \rho(x) \wedge \rho(y) \). For \( \sigma \in \mathbb{R}, v \in (0, 1], \) and \( k = 0, 1, 2, \ldots, \) as in [4], define
\[
[f]_k^{(\sigma)} = \sup_{x \in \Omega} \rho^{k+\sigma}(x)|D^\beta f(x)|,
\]
\[
[f]_{k+v}^{(\sigma)} = \sup_{x, y \in \Omega} \rho^{k+v+\sigma}(x, y)\frac{|D^\beta f(x) - D^\beta f(y)|}{|x - y|^v},
\]
\[
[f]_k^{(\sigma)} = \sum_{j=0}^k [f]_j^{(\sigma)}.
\]

Fix a constant \( \varepsilon_0 > 0 \). For \( \gamma \geq 0 \), define \( \gamma_+ = \gamma \) if \( \gamma \) is an integer, and \( \gamma_+ = \gamma + \varepsilon_0 \) otherwise.

Assumption 2.4.

(i) The functions \( a^{ij}(t, \cdot) \) are uniformly continuous in \( x \). In other words, for each \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that
\[
|a^{ij}(t, x) - a^{ij}(t, y)| \leq \varepsilon
\]
for all \( t \) and \( x, y \in \Omega \) whenever \( |x - y| < \delta \).

(ii) For any \( t > 0 \),
\[
|a^{ij}(t, \cdot)|^0_{\gamma_+} + |b^j(t, \cdot)|^0_{\gamma_+} + |c(t, \cdot)|^0_{\gamma_+} \leq K.
\]

Here are our main results.

Theorem 2.5. Let \( p \in [2, \infty), \gamma \in (0, \infty) \) and Assumptions 2.3 and 2.4 be satisfied. Then there exists \( \beta_0 = \beta_0(p, d, \Omega) > 0 \) so that if \( \theta \in (p - 2 + d - \beta_0, p - 2 + d + \beta_0) \) then for any \( f \in \psi^{-1-2\alpha} \|p, \theta\|_{p, \theta}(\Omega, T) \) and \( u_0 \in U^{\gamma_+} \|p, \alpha\|_{p, \alpha}(\Omega) \) Eq. (2.2) with initial data \( u_0 \) admits a unique solution \( u \) in the class \( S^{\gamma_+} \|p, \theta\|_{p, \alpha}(\Omega, T) \), and for this solution
\[
\|u\|_S^{\gamma_+} \leq C \left( \|\psi^{-1+2\alpha} f\|_{p, \theta}(\Omega, T) + \|u_0\|_{U^{\gamma_+}} \right),
\]
(2.11)

where \( C = C(d, p, \gamma, \theta, \delta_0, K, T, \Omega) \).

Lemmas 2.1(ii) and 2.2(i) easily yield the following result.

Corollary 2.6. Let \( u \in S^{\gamma_+} \|p, \theta\|_{p, \alpha}(\Omega, T) \) be the solution in Theorem 2.5.

(i) If \( \gamma + 2 - d/p = m + v \) for some \( m = 0, 1, \ldots, v \in (0, 1] \), and \( i, j \) are multi-indices such that \( |i| \leq m, |j| = m \), then for each \( t \)
\[
\psi^{|i|+\theta/p} D^i u \in C(\Omega), \quad \psi^{m+1+\theta/p} D^j u \in C^v(\Omega).
\]

In particular,
\[
|\psi^{|i|} D^i u(x) | \leq C \psi^{1-\theta/p}(x).
\]

(ii) Let \( 2/p < \beta \leq 1, \quad \gamma + 2 - \beta - d/p = k + \varepsilon \).

where \( k \in \{0, 1, 2, \ldots\} \) and \( \varepsilon \in (0, 1] \). Then for \( \delta := \beta(\alpha + 1) - 1 \) and multi-indices \( i, j \) such that \( |i| \leq k \) and \( |j| = k \),
sup_{s \neq t} \frac{|\psi^{s}|^{1+\theta/p} D^{s}(u(t) - u(s))|_{C(\Omega)}}{|t-s|^{\beta/2-1/p}} + sup_{s \neq t} \frac{|\psi^{s+t+\varepsilon/\theta} D^{s+t}(u(t) - u(s))|_{C^{s}(\Omega)}}{|t-s|^{\beta/2-1/p}} \\
\leq C \left( \| \psi^{1+2\alpha} f \|_{\|_{\|_{p,\theta,\alpha}(\Omega,T)}} + \| u_0 \|_{\|_{p,\theta,\alpha}(\Omega)} \right).

3. Proof of Theorem 2.5

We introduce some results to prove Theorem 2.5.

Lemma 3.1.

(i) Let $s = |\gamma|$ if $\gamma$ is an integer, and $s > |\gamma|$ otherwise, then

$$ \|au\|_{H_{p,\theta}^{|\gamma|}(\Omega)} \leq C(d,s,\gamma) |a|_{H_{p,\theta}^{|\gamma|}(\Omega)} \|u\|_{H_{p,\theta}^{|\gamma|}(\Omega)}. $$

(ii) If $\gamma = 0, 1, 2, \ldots$, then

$$ \|au\|_{H_{p,\theta}^{|\gamma|}(\Omega)} \leq C \sup_{\Omega} |a|_{H_{p,\theta}^{|\gamma|}(\Omega)} \|u\|_{H_{p,\theta}^{|\gamma|}(\Omega)} + C_0 |a|_{H_{p,\theta}^{|\gamma|}(\Omega)} \|u\|_{H_{p,\theta}^{|\gamma|}(\Omega)} $$

where $C_0 = 0$ if $\gamma = 0$.

(iii) If $0 \leq r \leq s$, then

$$ |a|_{r} \leq C(d,r,s) \left( \sup_{\Omega} |a| \right)^{1-r/s} (|a|_{s})^{r/s}. $$

Proof. For (i), see Theorem 3.1 in [15]. (ii) is an easy consequence of (2.6), and (iii) is from Proposition 4.2 in [14].

Remark 3.2. By Lemma 3.1, for any $\nu \geq 0$, $\psi^{\nu}$ is a point-wise multiplier in $H_{p,\theta}^{\gamma}(\Omega)$. Thus if $\theta_1 \leq \theta_2$ then

$$ \|u\|_{H_{p,\theta_1}^{\gamma}(\Omega)} \leq N \|\psi^{(\omega-\theta_1)/p} u\|_{H_{p,\theta_2}^{\gamma}(\Omega)} \leq N \|u\|_{H_{p,\theta_2}^{\gamma}(\Omega)}. $$

Lemma 3.3. Let $\{\xi_n\}$ be a sequence of $C_0^\infty(\Omega)$ functions such that

$$ |D^\mu \xi_n| \leq Ce^{\mu n}, \quad \text{supp} \xi_n \subset \{ x \in \Omega : e^{-n-k_0} < \rho(x) < e^{n+k_0} \} $$

for some $k_0 > 0$. Then for any $u \in H_{p,\theta}^{\gamma}(\Omega)$,

$$ \sum_{n} \| \xi_n(x) u(e^{n}x) \|_{H_{p}^{\gamma}(\Omega)} \leq C \|u\|_{H_{p,\theta}^{\gamma}(\Omega)} $$

If in addition

$$ \sum_{n} |\xi_n(x)| > \delta > 0, $$

then the reverse inequality also holds.

Proof. See Theorem 3.3 in [15].

Lemma 3.4. For any nonnegative integer $n \geq \gamma$, the set

$$ S_{p,\theta,\alpha}^n (\Omega,T) \cap \bigcup_{k=1}^\infty C([0,T],C_{0}^\infty(\Omega_k)), $$

where $\Omega_k := \{ x \in \Omega : \psi(x) > 1/k \}$, is dense in $S_{p,\theta,\alpha}^{\gamma}(\Omega,T)$.

Proof. It is enough to repeat the proof of Theorem 2.9 in [11], where the lemma is proved when $\alpha = 0$ and $\Omega \equiv \mathbb{R}^d_+$. 

Lemma 3.5. Assume that $a^{ij}$ is uniformly continuous in $x \in \mathbb{R}^d$ and
\[ |a^{ij}(t, \cdot)|_{C^{1,c}} \leq K. \]
Also let $f \in H^r_p(T)$, $u_0 \in U^{r+2}_p$ and $u \in H^{r+1}_p(T)$ be a solution of
\[ u_t = a^{ij}u_{x^i x^j} + f, \quad u(0, \cdot) = u_0. \]
Then (i) $u \in H^{r+2}_p(T)$ and
\[ \|u\|_{H^{r+2}_p(T)} \leq C \left(\|u\|_{H^{r+1}_p(T)} + \|f\|_{H^r_p(T)} + \|u_0\|_{U^{r+2}_p}\right), \tag{3.1} \]
where $C$ depends only on $d$, $p$, $\delta_0$, $K$ and the modulus of continuity of $a^{ij}$.

(ii) The term $\|u\|_{H^{r+1}_p(T)}$ in (3.1) can be dropped if the constant $C$ is allowed to depend also on $T$.

Proof. This is a well-known result. See, for instance, Lemma 6.6 and Theorem 5.1 in [8]. We only mention that some constants appearing in the proof of Lemma 6.6 and Theorem 5.1 in [8] depend also on $T$, but this dependency is just used to drop the them $\|u\|_{H^{r+1}_p(T)}$ in (3.1). \qed

In the following lemma, $p \in (1, \infty)$ and $\theta \in \mathbb{R}$.

Lemma 3.6. Let $a^{ij}$ be independent of $x$ and $b^i = c = 0$. If $f \in \psi^{-1-2\alpha} H^{r}_p(\Omega, T)$, $u_0 \in U^{r+2}_p(\Omega)$ and $u \in S^{r+1}_p(\Omega, T)$ is a solution of Eq. (2.2), then $u \in S^{r+2}_p(\Omega, T)$ and
\[ \|\psi^{-1}u\|_{H^{r+2}_p(\Omega, T)} \leq C \left(\|\psi^{-1}u\|_{H^{r}_p(\Omega, T)} + \|\psi^{1+2\alpha} f\|_{H^{r}_p(\Omega, T)} + \|u_0\|_{U^{r+2}_p}\right). \tag{3.2} \]

Proof. Denote $c_n = e^{2n(1+\alpha)}$. By Lemma 2.1(iii),
\[ \|\psi^{-1}u\|_{H^{r+2}_p(\Omega, T)} \leq C \sum_{n=-\infty}^{\infty} e^{n(p-2\alpha)} \|u(e^n x)\|_{\psi^{r+2}_p(\Omega, T)} = C \sum_{n=-\infty}^{\infty} e^{n(p-2\alpha)} \|u\|_{H^{r}_p(\gamma_1 T)}, \tag{3.3} \]
where $\gamma_n(x) := u(c_n t, e^n x) \zeta \in \psi^{-1}(c_n, d)$. Choose a nonnegative function $\eta \in C^\infty_0(c, d)$ so that $[c, d] \subset \mathbb{R}$ and $\eta = 1$ on the support of $\zeta$. Define $\eta_n(x) = \eta(e^n x)$. Then
\[ \eta_n \in C^\infty_0(\Omega), \quad |D^m \eta_n(x)| \leq C(\eta)e^{mn}. \]
Define
\[ a_n^{ij}(t, x) = e^{2\alpha n} \psi^{-2\alpha} (e^n x) \eta_n(e^n x) a^{ij}(c_n t) + (1 - \eta_n(e^n x)) \delta^{ij}. \]
Observe that on the support of $\eta_n(e^n x)$,
\[ e^{-\psi} (e^n x) \in [c, d], \quad e^{mn} \left| (D^{m+1} \psi)(e^n x) \right| \leq C(m) < \infty. \]
Thus one can easily check that there exists constant $c > 0$ so that
\[ c \delta_0 I \leq (a_n^{ij}) \leq c^{-1} K I, \quad \forall n, \]
\[ \sup_{n, t} |a_n^{ij}|_{C^k} \leq C(k) < \infty, \quad \forall k. \]
Note that $\gamma_n \in H^{r+1}_p(\gamma_1 T)$ satisfies
\[ (\gamma_n)_t = a_n^{ij} \gamma_n x^i x^j + f_n, \]
where
\[ f_n(t, x) = -2e^{2\alpha n} a_n^{ij}(t, x) u_\psi(c_n t, e^n x) \zeta_{-n x^i} (e^n x) = -2e^{2\alpha n} a_n^{ij}(t, x) u(c_n t, e^n x) \zeta_{-n x^i} (e^n x) + e^{2\alpha n + 2\alpha} f(c_n t, e^n x) \zeta_{-n x^i} (e^n x). \]
It is easy to check $f_n \in H^{r+2}_p(\gamma_1 T)$. Thus by Lemma 3.5, we have $\gamma_n \in H^{r+2}_p(\gamma_1 T)$ and
\[ \|\gamma_n\|_{H^{r+2}_p(\gamma_1 T)} \leq C \left(\|\gamma_n\|_{H^{r+1}_p(\gamma_1 T)} + \|f_n\|_{H^{r+2}_p(\gamma_1 T)} + \|u_0\|_{U^{r+2}_p}\right). \]
where \( C = C(d, p, \gamma, \delta_0, K, \eta) \). Next we apply Lemma 3.3 with \( \xi_n = e^{-n\delta_0} \) or \( \xi_n = e^{-2\eta} \) and get

\[
\sum_{n=-\infty}^{\infty} e^{n(\theta-p+2+2\alpha)} \| f_n \|_{L_p}^p \leq C \sum_{n} e^{n\| u \|_{L_p}^p} \| (e^{n\xi}e^{n\xi})(e^{n\xi}) \|_{L_p}^p + C \sum_{n} e^{n(\theta-p)} \| (e^{n\xi}e^{n\xi})(e^{n\xi}) \|_{L_p}^p + C \sum_{n} e^{n(\theta+p+2\alpha)} \| f(e^{n\xi}) \|_{L_p}^p \leq C \| u \|_{L_p}^p \| (\Omega, T) + C \| u \|_{L_p, p, \beta, \alpha}(\Omega, T) + C \| f \|_{L_p, p, \beta, \alpha}(\Omega, T).
\]

Also,

\[
\sum_{n=-\infty}^{\infty} e^{n(\theta-p+2+2\alpha)} \| v_n(0) \|_{L_p}^p \leq \| u_0 \|_{L_p, \beta, \alpha}(\Omega).
\]

Now to get (3.2), it is enough to use

\[
\| u \|_{H^p, \beta}(\Omega) + \| u \|_{H^p, p, \beta}(\Omega) \leq C \| \psi^{-1} u \|_{H^p, \beta}(\Omega),
\]

and solution \( u \)

\[
\sum_{n=-\infty}^{\infty} e^{n(\theta-p+2+2\alpha)} \| v_n(0) \|_{L_p}^p \leq \| u_0 \|_{L_p, \beta, \alpha}(\Omega).
\]

Lemma 3.7. Let \( a^{ij} \) be independent of \( x \) and \( b^i = c = 0 \). Then for any solution \( u \in \mathcal{S}_{p, p-2+\alpha}(\Omega, \Delta) \) of (2.2) we have

\[
\| u \|_{\mathcal{S}_{p, p-2+\alpha}(\Omega, \Delta)} \leq C \| \psi^{1/2} f \|_{L_p, p, \beta, \alpha}(\Omega, \Delta) + C \| u(0) \|_{L_p, p, \beta, \alpha}(\Omega),
\]

where \( C = C(d, p, \Omega) \).

Proof. Due to Lemma 3.4, we may assume that \( u \) is sufficiently smooth in \( x \) and vanishes near \( \partial \Omega \). Since \( \| u \|^p \chi = p|u|^{p-2}uu_1 \),

\[
|u(T)|^p \psi^{2\alpha} = \psi^{2\alpha} |u(0)|^p + p \int_0^T |u|^{p-2} u(a^{ij}u_{xj}) + f \psi^{2\alpha} dt.
\]

By (2.9) and integration by parts,

\[
\delta_0(p-1) \int_0^T \int_\Omega |u|^{p-2}|Du|^2 \, dx \, dt \leq p(p-1) \int_\Omega |u|^{p-2}a^{ij}u_{xj} \, dx \, dt
\]

\[
\leq \int_\Omega \| \psi^{1/2} \|_{L_p, \beta, \alpha}(\Omega, \Delta) \| u(0) \|_{L_p, p-2+\alpha}(\Omega) + C \| \psi^{1/2} f \|_{L_p, p, \beta, \alpha}(\Omega, \Delta).
\]

Now we apply inequality (1.1) with \( g = |u|^{p/2} \) to get

\[
\int_\Omega \int_\Omega |\psi^{-1} u|^p \psi^{p-2} \, dx \, dt \leq C \int_\Omega \int_\Omega |u|^{p-2}|Du|^2 \, dx \, dt.
\]

Combining (3.5) and (3.6) we get

\[
\| \psi^{-1} u \|_{L_p, p-2+\alpha}(\Omega, \Delta) \leq C \| u(0) \|_{L_p, p-2+\alpha}(\Omega) + C \| \psi^{1+2\alpha} f \|_{L_p, p, \beta, \alpha}(\Omega, \Delta).
\]

This and (3.2) easily yield (3.4). □

Lemma 3.8. Let \( a^{ij} \) be independent of \( x \) and \( b^i = c = 0 \). Then there exists \( \beta_1 = \beta_1(d, p, \Omega) > 0 \) so that for any \( \theta \in (d + p - 2 - \beta_2, d + p - 2 + \beta_1) \) and solution \( u \in \mathcal{S}_{p, p, \beta}(\Omega, \Delta) \) of (2.2) we have

\[
\| \psi^{-1} u \|_{H^p, \beta}(\Omega) \leq C \| \psi^{1+2\alpha} f \|_{L_p, p, \beta, \alpha}(\Omega, \Delta) + C \| u(0) \|_{L_p, p, \beta, \alpha}(\Omega),
\]

where \( C = C(d, p, \Omega) \).
**Proof.** Let $v = (-\theta + p - 2 + d)/p$, $v = \psi^nu$, then by Lemma 2.1, $v \in H^{2}_{p,\beta}((\Omega, T))$ and

$$v_t = \psi^{-2a}v + f\psi^nu + \psi^{-1-2a}\psi\tilde{f},$$

where

$$\tilde{f} := \psi\psi_{u_x} + [(v-1)|D\psi|^2 + \psi\Delta\psi]\psi^{-1}u.$$

By Lemmas 2.1 and 3.7,

$$\|\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)} \leq C\|\psi\psi_{u_x} + [(v-1)|D\psi|^2 + \psi\Delta\psi]\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)}.$$

Since $\psi, \psi, \psi\Delta\psi$ are bounded, estimate (3.7) follows if $v$ is sufficiently close to zero. The lemma is proved. 

Now we prove a priori estimate.

**Lemma 3.9.** Let assumptions in Theorem 2.5 hold with $\beta_0 = \beta_1$ taken from Lemma 3.8, then estimate (2.11) holds given that a solution already exists.

**Proof.** Step 1. Assume

$$|a^{ij}(t, x) - a^{ij}(t, y)| + |b^{ij}(t, x)| + |c(t, x)| \leq \kappa, \quad \forall t, x, y.$$

We prove that there exists $\kappa_0 = \kappa_0(d, \gamma, \theta, \delta_0, K) > 0$ so that the assertion of the lemma holds if $\kappa \leq \kappa_0$. Fix $x_0 \in \Omega$ and denote $a^{ij}_0(t, x) = a^{ij}(t, x_0).$ Then $u$ satisfies

$$u_t = \psi^{-2a}a^{ij}_0u_{x^i\bar{k}} + \psi^{-1-2a}f_t + f,$$

where

$$f = (a^{ij} - a^{ij}_0)u_{x^i\bar{k}} + b^{ij}u_{x^i\bar{k}} + c\psi^{-1}u.$$

By Lemmas 3.6 and 3.8,

$$\|\psi^{\kappa}u\|_{H^{1}_{p,\beta}(\Omega, T)} \leq C\|\psi\tilde{f}\|_{H^{1}_{p,\beta}(\Omega, T)} + C\|\psi^{\kappa}u\|_{H^{2}_{p,\beta}(\Omega, T)}.$$

If $\gamma' \neq 1, 2, 3, \ldots$, then by Lemma 3.1

$$\|\psi^{\kappa}u\|_{H^{2}_{p,\beta}(\Omega, T)} \leq C\|\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)}.$$

Thus it is enough to take $\kappa_0$ so that $C_1\kappa^{\gamma'} < 1/2$ for all $\kappa \leq \kappa_0$. If $\gamma = 1, 2, 3, \ldots$, then by Lemma 3.1

$$\|\psi^{\kappa}u\|_{H^{2}_{p,\beta}(\Omega, T)} \leq C\|\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)} + C\|\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)}.$$

Similarly,

$$\|\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)} \leq 2C_2\|\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)} + C\|\psi^{-1}u\|_{H^{2}_{p,\beta}(\Omega, T)}.$$

Take $\kappa_0 = \kappa_0(0)$ chosen in the above when $\gamma = 0$. Then it suffices to take $\kappa_0 = \kappa_0(\gamma')$ so that $\kappa_0 < \kappa_0(0) \wedge C_2/4$. 

Step 2. Let $x_0 \in \partial\Omega$. Fix a nonnegative function $\mu \in C_i^\infty(B_1(0))$ so that $\mu(x) = 1$ for $|x| \leq 1/2$ and define

$$n_\mu(x) = \mu(n(x - x_0)), \quad a_\mu(t, x) = a(t, x)\mu_\mu(x) + (1 - n_\mu(x))a(t, x_0),$$

$$b_\mu = b\mu_n, \quad c_\mu = c\mu_n.$$

Then it is easy to check...
where
\[ \psi \]

Lemma 3.10. Thus (2.11) follows from this and Gronwall’s inequality.

By Lemmas 2.1 and 3.1,
\[ \| v \|_{H^1(\Omega, t)} \leq C \| \psi \|_{L^2(\Omega, t)} + \| \psi \|_{H^1(\Omega, t)} + \| u \|_{H^1(\Omega, t)} + \| u \|_{H^2(\Omega, t)} \].
By Lemmas 2.1 and 3.1,
\[ \| \psi \|_{L^2(\Omega, t)} \leq C \| \psi \|_{H^1(\Omega, t)} + C \| \psi \|_{H^1(\Omega, t)} \leq C \| u \|_{H^1(\Omega, t)} \]

Consequently,
\[ \| v \|_{H^1(\Omega, t)} \leq C \left( \| \psi \|_{L^2(\Omega, t)} + \| \psi \|_{H^1(\Omega, t)} + \| u \|_{H^2(\Omega, t)} \right). \]

Now to estimate \( u \), one introduces a partition of unity \( \zeta_i \), \( i = 0, 1, \ldots, N \) so that \( \zeta_0 \in C^\infty_0(\Omega) \) and \( \zeta_i - \mu(2n(x-x_i)) \), \( x_i \in \partial \Omega \) for \( i \geq 1 \). Then one estimates \( u \zeta_0 \) using Lemma 3.5(ii) and others as above. By summing up the norms and using Lemma 2.2(ii) one gets
\[ \| u \|_{H^2(\Omega, t)} \leq C \left( \| \psi \|_{L^2(\Omega, t)} + \| \psi \|_{H^1(\Omega, t)} + \| u \|_{H^2(\Omega, t)} \right). \]

Thus (2.11) follows from this and Gronwall’s inequality. \( \square \)

Lemma 3.10. Let \( f \in L_p((0, T], C^k_0(\Omega)) \) for some \( k > 0 \) with its first derivatives in \( t, x \) bounded. Then the equation
\[ u_t = \psi^{−2\alpha} \Delta u + f, \quad u(0) = 0 \]

has a solution \( u \in H_{0}^{2}(\Omega, T) \).

Proof. By Lemma 3.6 we only need to prove that there exists a solution \( u \in H_{0}^{2}(\Omega, T) \). Let \( n > k \). Since \( \partial \Omega \in C^\infty \) and \( \psi^{−2\alpha} \) is bounded and infinitely differentiable in \( \Omega \), by Theorem 2.10 in [6] (cf. Theorem IV 5.2 in [13]), there is a unique (classical) solution \( u^n \in H_{0}^{2}(\Omega, T) \) of
\[ u^n_t = \psi^{−2\alpha} \Delta u^n + f, \quad u^n(0, \cdot) = 0. \]
such that \( u^n|_{\partial \Omega} = 0 \) and \( D^n \), \( D^2 \) are bounded in \( [0, T] \times \Omega \). Define \( u^n(x) = 0 \) for \( x \notin \Omega \), then \( u^n \) is Lipschitz continuous in \( \Omega \).

From
\[ |u^n(T, x)|^2 = \int_0^T [2u^n \Delta u^n + 2u^n f \psi^{2\alpha}] dt, \quad \forall x \in \Omega \]

we get
\[ 2 \int_0^T \int_\Omega |D^n|^2 \, dx \, dt = 2 \int_0^T \int_\Omega \left| \psi^{−1} u^n \right|^2 \, dx \, dt + e^{-1} \int_0^T \int_\Omega |\psi^{1+2\alpha} f|^2 \, dx \, dt. \]
By Hardy inequality,
\[
\sup_n \left( \left\| \psi^{-1} u^n \right\|_{L^2(\varOmega, T)} + \left\| Du^n \right\|_{L^2(\varOmega, T)} \right) < \infty.
\]
Now we choose \( \xi^n \in C_0^\infty(\varOmega_k) \) such that \( \xi^n = 1 \) on \( \varOmega_k \), \( \rho \xi^n, \rho^2 \xi^n \) are bounded in \( \varOmega \) and \( \xi^n(x) \to 1 \) for \( x \in \varOmega \) as \( n \to \infty \). Then \( u^n \xi^n \in \mathcal{S}^{2,\varphi}_{2,\varphi,0}(\varOmega, T) \) satisfies
\[
\left( u^n \xi^n \right)_t = \psi^{-2a} \Delta (u^n \xi^n) - 2\psi^{-2a} u^n \xi^n_x - \psi^{-2a} u^n \xi^n_{xx} + f.
\]
By Lemma 3.9,
\[
\left\| u^n \xi^n \right\|_{H^2(\varOmega, T)} \leq C \left\| u^n \xi^n_x \right\|_{L^2(\varOmega, T)} + C \left\| \psi^1 \right\| \right\|_{L^2(\varOmega, T)}.
\]
By dominated convergence theorem,
\[
\left\| u^n \xi^n \right\|_{H^2(\varOmega, T)} + \left\| \psi^{-1} \psi^2 \left( \xi^n \right) \right\|_{L^2(\varOmega, T)} \to 0 \quad \text{as} \quad n \to \infty.
\]
Denote \( v^n = u^n \xi^n \in \mathcal{S}^{2,\varphi}_{2,\varphi,0}(\varOmega, T) \), then \( \{v^n\} \) is a bounded sequence in \( \mathcal{S}^{2,\varphi}_{2,\varphi,0}(\varOmega, T) \). To prove that there exists \( u \in \mathcal{S}^{2,\varphi}_{2,\varphi,0}(\varOmega, T) \), the weak limit of \( v^n \), such that \( u \) is a solution of (3.8), it is enough to repeat the proof of Theorem 3.11 in [8]. One difference is that one has to use (2.7) in our article instead of inequality (3.4) in [8]. Now we proceed as in the proof of Lemma 3.7 and get
\[
\int_0^T \int_\varOmega \left\| \psi^{-1} u^n \right\|_p \psi^{p-2} dx dt \leq C \int_0^T \int_\varOmega \left\| u^n \right\|^{p-2} |Du^n|^2 dx dt = C \int_0^T \int_\varOmega \left\| u^n \right\|^{p-2} |Du^n|^2 dx dt \leq C \left\| \psi \right\|_{L^p(\varOmega, T)} \right\|_{L^p(\varOmega, T)}.
\]
Thus
\[
\sup_n \left\| \psi^{-1} u^n \right\|_{L^p(\varOmega, T)} < \infty,
\]
and we conclude that \( u \in \mathcal{S}^{2,\varphi}_{2,\varphi,0}(\varOmega, T) \). The lemma is proved. \( \square \)

**Proof of Theorem 2.5.** The a priori estimate from Lemma 3.9 combined with the method of continuity show that it only remains to prove solvability of the equation
\[
u_t = \psi^{-2a} \Delta u + f, \quad u(0, \cdot) = u_0.
\]
Since \( C_0^\infty(\varOmega) \) is dense in \( U_0^{p, \varphi, \varphi, 0}(\varOmega) \), it suffices to concentrate on \( u_0 \in C_0^\infty(\varOmega) \). Then passing to \( u \) to \( u - u_0 \) we see that we may assume \( u_0 = 0 \). Similarly we may assume that \( f \) is bounded on \( \varOmega \times [0, T] \) along with each derivative in \((t, x)\) and vanishes if \( x \) is in a neighborhood of the boundary of \( \varOmega \). In this case, due to Lemma 3.6 we only need to show the existence of solution in the space \( \mathcal{S}^{2,\varphi}_{2,\varphi,0}(\varOmega, T) \). Thus, by Lemma 3.10 and Remark 3.2, the theorem is proved if \( \theta \in \{d + p - 2, d + p - 2 + \beta_1\} \). Take \( k_0 \) from Step 1 in proof of Lemma 3.9 corresponding to the case when \( \theta = p - 2 + d \). Then by inspecting the proof of the previous lemmas, one easily checks that for any \( \beta \geq 0 \) the equation
\[
u_t = \psi^{-2a} \Delta u + \psi^{-1 - 2a} \beta u_x + \psi^{-2 - 2a} \beta u + \psi^\beta f
\]
has a solution \( u \in \mathcal{S}^{2,\varphi}_{p, p - 2 + d, \varphi}(\varOmega, T) \) if, instead of (2.10),
\[
|b^t| + |c| < k_0, \quad \forall t, x.
\]
Define
\[
b^t = -2\beta \psi_x, \quad c = -\beta(\beta - 1)\Delta \psi^2 + \beta \psi \Delta \psi.
\]
and choose \( \bar{\beta}_0 \) so that for any \( \beta < \bar{\beta}_0, \) \( |b^t| + |c| < k_0 \). Observe that \( v := \psi^{-\beta} u \in \mathcal{S}^{2,\varphi}_{p, p - 2 + d, \varphi}(\varOmega, T) \) satisfies
\[
v_t = \psi^{-2a} \Delta v + f.
\]
Thus to finish the proof of the theorem it is enough to take \( \beta_0 := \bar{\beta}_0 \wedge \beta_1 \). Actually the theorem is true for any \( \theta \in \{p - 2 + d - \beta_1, p - 2 + d + \beta_1\} \). This can be proved by showing \( k_0, \) as a function of \( \theta, \) is bounded away from zero on any closed subset of \( \{p - 2 + d - \beta_1, p - 2 + d + \beta_1\} \) and repeating the above process starting from \( \theta = p - 2 + d - np \bar{\beta}_0 \) instead of from \( \theta = p - 2 + d \) until \( p - 2 + d - (n + 1)p \bar{\beta}_0 \). The theorem is proved. \( \square \)

We finish the article with the following remark.
Remark 3.11.

(i) In this article we only consider bounded domains, however actually, Theorem 2.5 is true for any unbounded domains allowing inequality (1.1) provided $\psi$ is chosen so that $\psi$ is "bounded" and $\rho(x) \sim \psi(x)$ near $\partial \Omega$.

(ii) To estimate $\| \psi^{-1} u \|_{L^p(\Omega,T)}$ in Lemma 3.7 we used integration by parts and Hardy inequality. If $\Omega$ is sufficiently smooth, say $\Omega = \mathbb{R}^d_+$, then by using Corollary 6.2 in [10] instead of Hardy inequality, one can prove that Lemma 3.7 holds for the wider range of weights. Actually it holds for any $d - 1 < \theta < d - 1 + p$.

(iii) The main feature of this article is that in this article the leading coefficients are assumed to blow up near the boundary at the rate of $\rho^{-2\alpha}(x), \alpha \in [0, \infty)$ and $\partial \Omega$ is allowed to be a non-Lipschitz domain as long as inequality (1.1) holds. Furthermore the weighted $L_p$-norms of derivatives of solutions of any order are obtained along with the weighted Hölder estimates of solutions with respect to time and space variables.

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References