ZERO PRODUCTS OF TOEPLITZ OPERATORS
WITH n-HARMONIC SYMBOLS

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ABSTRACT. On the Bergman space of the unit polydisk in the complex n-space, we solve the zero-product problem for two Toeplitz operators with n-harmonic symbols that have local continuous extension property up to the distinguished boundary. In the case where symbols have additional Lipschitz continuity up to the whole distinguished boundary, we solve the zero-product problem for products with four factors. We also prove a local version of this result for products with three factors.

1. INTRODUCTION

Let $D$ be the unit disk in the complex plane. For a fixed positive integer $n$, the unit polydisk $D^n$ is the cartesian product of $n$ copies of $D$. Let $L^2 = L^2(D^n, V)$ denote the usual Lebesgue space where $V = V_n$ is the volume measure on $D^n$ normalized to have total mass 1. We let $A^2 = A^2(D^n)$ denote the Bergman space consisting of all holomorphic functions in $L^2$. Due to the mean value property of holomorphic functions, the Bergman space $A^2$ is a closed subspace of $L^2$, and thus is a Hilbert space.

Since every point evaluation is a bounded linear functional on $A^2$, there corresponds to every $a \in D^n$ a unique function $K_a \in A^2$ which has the following reproducing property:

\begin{equation}
    f(a) = \int_{D^n} f(z) \overline{K_a(z)} \, dV(z)
\end{equation}

for $f \in A^2$. The function $K_a$ is the well-known Bergman kernel and its explicit formula is given by

$$K_a(z) = \prod_{j=1}^{n} \frac{1}{(1 - \overline{a_j} z_j)^2}, \quad z \in D^n;$$

see, for example, Proposition 1.4.24 of [13]. Here, and elsewhere, $z_j$ denotes the $j$-th component of $z$. It follows from (1.1) that the Hilbert space orthogonal projection $P$ from $L^2$ onto $A^2$, called the Bergman projection, is realized as an integral operator

$$P\psi(a) = \int_{D^n} \psi(z) \overline{K_a(z)} \, dV(z)$$

for $\psi \in L^2$.

For a function $u \in L^\infty$, the Toeplitz operator $T_u$ with symbol $u$ is defined by

$$T_u f = P(uf)$$

for $f \in A^2$. It is clear that $T_u : A^2 \to A^2$ is a bounded linear operator.
In this paper we study the zero-product problem of whether the zero-product of several Toeplitz operators has only the trivial solution. More explicitly, the problem we consider is

Does \( T_{u_1} \cdots T_{u_N} = 0 \) imply that some \( u_j \) is identically zero?

This problem was first studied for Toeplitz operators on the Hardy space. In [5], Brown and Halmos actually studied a more general problem on the Hardy space of the unit disk and proved that \( T_u T_v = T_{\phi} \) if and only if either \( u \) or \( v \) is holomorphic and \( \phi = uv \). As an immediate consequence of their result, they easily derived that the zero-product problem with two factors has only the trivial solution. Later, on the same context of Hardy space of the unit disk, the zero-product problem has been solved by Guo [12] for products with five factors and by Gu [11] for products with six factors. Recently, Ding [10] solved this problem for products with two factors on the Hardy space of the polydisk. There has been no progress on higher dimensional balls, as far as we know.

For the Bergman space case, the study of the zero-product problem has begun only recently. On the setting of the Bergman space of the unit disk, Ahern and Ćučković [2] solved the zero-product problem for two factors with harmonic symbols. More recently, Ahern [1] gave another more general approach to solve the same problem. Recall that bounded measurable functions on the unit circle can be identified with boundary values of bounded harmonic functions on the unit disk. So, it seems quite natural (to us) to work with harmonic symbols as in [2] as substitutes for general symbols in the Hardy space case. However, the work in [2] shows that the Bergman space case, even with such harmonic symbols, is much more subtle than the Hardy space case. Nevertheless, there have been a couple of progresses for symbols other than harmonic ones. Ćučković [9] obtained some partial results when only one symbol is harmonic. Also, Ahern and Ćučković [3] solved the zero-product problem for two factors with radial symbols. Of course, the zero-product problem with general symbols even for two factors, remains still very far from its solution.

The higher dimensional cases have been also studied on the ball and polydisk. Recently, the polydisk case was solved by Choe et al. [8] for two factors with pluriharmonic symbols by extending the method in [2]. More recently, on the setting of the unit ball, the first two authors of the present paper [7] used an entirely different method to solve the zero-product problem for two factors with harmonic symbols that have local continuous extension property up to the boundary. At the same paper, they also solved the problem for multiple products with number of factors depending on the dimension in the case where symbols have additional (global or local) Lipschitz continuity up to the boundary. In this paper, we study the same problem on the polydisk.

Going from the ball to the polydisk, we need to adjust our setting suitable for the polydisk. According to our results below, it turns out that \( n \)-harmonic symbols on the polydisk are the right substitutes for harmonic symbols on the ball and the distinguished boundary of the polydisk is the right substitute for the boundary of the ball. Recall that a function \( u \in C^2(D^n) \) is called \( n \)-harmonic as in [14] if \( u \) is harmonic in each variable separately. More explicitly, \( u \) is \( n \)-harmonic if

\[
\partial_j \overline{\partial}_j u = 0, \quad j = 1, 2, \ldots, n
\]

where \( \partial_j = \partial / \partial z_j \) denotes the complex differentiation with respect to \( z_j \). Also, recall that the distinguished boundary of \( D^n \) is \( T^n \), the cartesian product of \( n \) copies of the unit circle \( T = \partial D \).

First, we consider \( n \)-harmonic symbols with local continuous extension property up to the distinguished boundary \( T^n \) and solve the zero-product problem for two factors. In what
follows, we let $h^\infty$ denote the class of all bounded $n$-harmonic functions on $D^n$. Also, a “boundary open” set refers to a relatively open subset of $T^n$. The following is our first result. In case of the disk, this result coincides with Theorem 1.1 of [7], which in turn is contained in the work of Ahern and Ćučković [2] mentioned above.

**Theorem 1.1.** Suppose that $u_1, u_2 \in h^\infty$ are continuous on $D^n \cup W$ for some boundary open set $W$. If $T_{u_1}T_{u_2} = 0$, then $u_1 = 0$ or $u_2 = 0$.

Next, we consider the case where symbols have additional Lipschitz continuity up to the whole or some part of the distinguished boundary. Such extra Lipschitz continuity enables us to extend our method to products with three or four factors. Given a subset $X \subset \mathbb{C}^n$, recall that the Lipschitz class of order $\epsilon \in (0, 1]$, denoted by $Lip_\epsilon(X)$, is the class of all functions $f$ on $X$ such that $|f(z) - f(w)| = O(|z - w|^{\epsilon})$ for $z, w \in X$.

Our second result solves the zero-product problem with four factors when symbols are $n$-harmonic and have global Lipschitz continuity up to the distinguished boundary. We let $\hat{D}^n = D^n \cup T^n$.

**Theorem 1.2.** Let $u_1, u_2, u_3, u_4 \in Lip_\epsilon(U \cap \hat{D}^n) \cap h^\infty$ for some $\epsilon > 0$ and for some open set $U$ containing $T^n$. If $T_{u_1}T_{u_2}T_{u_3}T_{u_4} = 0$, then $u_j = 0$ for some $j$.

For $n$-harmonic symbols that have only local Lipschitz continuity up to the distinguished boundary, we can also apply the proof of Theorem 1.2 to solve the zero-product problem, but with only three factors.

**Theorem 1.3.** Let $u_1, u_2, u_3 \in Lip_\epsilon(U \cap \hat{D}^n) \cap h^\infty$ for some $\epsilon > 0$ and for some open set $U$ with $U \cap T^n \neq \emptyset$. If $T_{u_1}T_{u_2}T_{u_3} = 0$, then $u_j = 0$ for some $j$.

While the main idea of our method of proofs is adapted from [7], substantial amount of unexpected analysis is required to overcome some different nature of the polydisks being product domains. For example, a uniqueness theorem (Proposition 2.2) and an example that follows reveal more involved nature of the polydisks compared with the balls. The whole part of later arguments is thus necessarily effected in a more complicated direction by such an unexpected uniqueness result.

Boundary continuity and $n$-harmonicity hypotheses in the theorems above play key roles in our arguments of the present paper. First, boundary continuity ensures (Proposition 2.1) that the Berezin transform of products of Toeplitz operators under consideration recovers the products of corresponding symbols at every distinguished boundary point where the symbols have continuous extensions. Also, both hypotheses allow us to use a uniqueness theorem (Proposition 2.2) for $n$-harmonic functions, which can be viewed as a product version of the local Hopf lemma obtained in [6]. In addition, it provides us quite explicit information on local behavior of symbol functions near a distinguished boundary vanishing point (Lemma 3.1). We do not know whether either boundary regularity or $n$-harmonicity can be removed in the hypotheses of our theorems above when $n \geq 2$. We do not know whether the number of factors are best possible under the given hypotheses, either.

In Section 2, we collect a couple of key facts which play the role of the starting point of our proofs. In Section 3, we prove Theorems 1.1, 1.2 and 1.3.

### 2. Preliminaries

In this section we prove two main ingredients of our arguments which might be of independent interests. One is a certain continuous extension property of the Berezin transform.
and the other is a uniqueness theorem for \( n \)-harmonic functions. These two propositions are what led us to require boundary regularity and \( n \)-harmonicity hypotheses in our theorems.

Let \( a \in D^n \) denote an arbitrary point, unless otherwise specified. Recall that the well-known Berezin transform of a bounded linear operator \( L \) on \( A^2 \) is a function \( \tilde{L} \) on \( D^n \) defined by

\[
\tilde{L}(a) = \int_{D^n} Lk_a(z)\overline{k_a(z)} \, dV(z)
\]

where \( k_a \) is the normalized kernel, namely,

\[
k_a(z) = \prod_{j=1}^{n} \frac{1 - |a_j|^2}{(1 - \overline{a}_jz_j)^2}, \quad z \in D^n.
\]

It is not hard to see that \( \tilde{L} \) is continuous on \( D^n \). In our application \( L \) will be the products of Toeplitz operators as in the hypotheses of theorems stated in the Introduction. Such operators \( L \) have additional properties useful for our purpose: \( \tilde{L} \) has the same amount of boundary continuity as symbols and the boundary value of \( \tilde{L} \) is precisely the product of symbols. In order to prove this, we first recall some well-known facts.

We let \( \varphi_a(z) = (\phi_a(z_1), \ldots, \phi_a(z_n)) \) where each \( \phi_a \) is the usual Möbius map on \( D \) given by

\[
\phi_a(z_j) = \frac{a_j - z_j}{1 - \overline{a}_jz_j}, \quad z_j \in D.
\]

The map \( \varphi_a \) is an automorphism on \( D^n \) such that \( \varphi_a \circ \varphi_a = \text{id} \). We define a linear operator \( U_a \) on \( A^2 \) by

\[
U_a f = (f \circ \varphi_a)k_a
\]

for \( f \in A^2 \). Since the real Jacobian of \( \varphi_a(z) \) is \( \prod_{j=1}^{n} |\phi'_a(z_j)|^2 = |k_a(z)|^2 \), we see that each \( U_a \) is an isometry on \( A^2 \). Also, since \( U_a k_a = (k_a \circ \varphi_a)k_a = 1 \), we have \( U_a U_a = I \) and thus \( U_a^{-1} = U_a \). Now, being an invertible linear isometry, \( U_a \) is unitary. Also, we have

\[
U_a T_u U_a = T_{u \circ \varphi_a};
\]

this is well known on the disk (see, for example, [4]) and the same proof works on the polydisk.

**Proposition 2.1.** Suppose that functions \( u_1, \ldots, u_N \in h^\infty \) are continuous on \( D^n \cup \{ \zeta \} \) for some \( \zeta \in T^n \). Let \( L = T_{u_N} \cdots T_{u_1} \). Then \( \tilde{L} \) continuously extends to \( D^n \cup \{ \zeta \} \) and \( \tilde{L}(\zeta) = (u_1 \cdots u_N)(\zeta) \).

The proof below is similar to that of the ball case (Proposition 2.1 of [7]) and is included here for reader’s convenience.

**Proof.** Let \( a \in D^n \). Note that \( k_a = U_a1 \). Also, by (2.1), we note that

\[
U_a L U_a = (U_a T_{u_N} U_a) \cdots (U_a T_{u_1} U_a) = T_{u_N \circ \varphi_a} \cdots T_{u_1 \circ \varphi_a},
\]

because \( U_a U_a = I \). Since \( U_a^* U_a = U_a^{-1} = U_a \), it follows that

\[
\tilde{L}(a) = \langle U_a L U_a 1, 1 \rangle = \langle T_{u_N \circ \varphi_a} \cdots T_{u_1 \circ \varphi_a} 1, 1 \rangle
\]

where \( \langle , \rangle \) denotes the inner product on \( L^2 \). We claim that, as \( a \to \zeta \), we have

\[
T_{u_N \circ \varphi_a} \cdots T_{u_1 \circ \varphi_a} 1 \to (u_N \cdots u_1)(\zeta) \quad \text{in} \quad L^2
\]

which, together with (2.2), implies the proposition.
Now, we prove the claim. Let \( a \to \zeta \). Then, for a given function \( u \) continuous on \( D^n \cup \{ \zeta \} \), we observe that \( \varphi_a \to \zeta \) pointwise (in fact, uniformly on compact sets) and thus \( u \circ \varphi_a \to u(\zeta) \) in \( L^2 \) by the dominated convergence theorem. In particular, we have \( P(u \circ \varphi_a) \to u(\zeta) \) in \( L^2 \). So, we see that the claim holds for \( N = 1 \). We now proceed by induction on \( N \). Assume that (2.3) holds for some \( N \geq 1 \) and consider the case of \( N + 1 \). Having (2.3) as induction hypothesis and denoting the \( L^p \)-norm by \( \| \cdot \|_p \), we have

\[
\| T_{u_{N+1} \circ \varphi_a} T_{u_N \circ \varphi_a} \cdots T_{u_1} 1 - (u_{N+1} u_N \cdots u_1)(\zeta) \|_2 \\
\leq \| T_{u_{N+1} \circ \varphi_a} T_{u_N \circ \varphi_a} \cdots T_{u_1} 1 - (u_N \cdots u_1)(\zeta) \|_2 \\
+ \| (u_N \cdots u_1)(\zeta) \| T_{u_{N+1} \circ \varphi_a} 1 - u_{N+1}(\zeta) \|_2 \\
\leq \| u_{N+1} \|_\infty \| T_{u_N \circ \varphi_a} \cdots T_{u_1} 1 - (u_N \cdots u_1)(\zeta) \|_2 \\
+ \| (u_N \cdots u_1)(\zeta) \| T_{u_{N+1} \circ \varphi_a} 1 - u_{N+1}(\zeta) \|_2 \\
\to 0
\]

so that (2.3) also holds for \( N + 1 \). This completes the induction and the proof of the proposition. \( \square \)

Now, we turn to a uniqueness property of bounded \( n \)-harmonic functions. We first recall a certain extension property of bounded \( n \)-harmonic functions across the boundary. So, let \( u \in H^\infty \) be an arbitrary function. As is well known, the radial limit \( u^*=\lim_{r\to 1} u(r\zeta) \) exists at almost all points \( \zeta \in T^n \) and \( u \) is recovered by the Poisson integral of \( u^* \):

\[
u(z) = \int_{T^n} u^*(\zeta) \prod_{j=1}^n \frac{1 - |z_j|^2}{|1 - \zeta_j|^2} \, d\sigma(\zeta), \quad z \in D^n
\]

where \( d\sigma \) is the Haar measure on \( T^n \). See [14, p. 31] for details. Thus, if \( u \) is in addition continuous on \( D^n \cup W \) and vanishes on \( W \) for some boundary open set \( W \), then it is easily seen from the Poisson integral formula above that \( u \) extends \( n \)-harmonically across \( W \).

We also need various notation. Working with \( n \)-harmonic functions, it seems quite natural and necessary to consider relevant differential operators componentwise. Recall \( \partial_j = \partial/\partial z_j \). Now, given \( j = 1, \ldots, n \), we let

\[
R_j = z_j \partial_j + \zeta_j \overline{\partial_j} \\
T_j = \sqrt{-1}(z_j \partial_j - \zeta_j \overline{\partial_j}) \\
\Delta_j = 4 \partial_j \overline{\partial_j}
\]

denote the \( j \)-th radial differential operator, the \( j \)-th real tangential differential operator and the \( j \)-th Laplacian, respectively. Here, and elsewhere, we abuse the notation \( \zeta_j \) for the function \( z \mapsto \zeta_j \). Then it is straightforward to see that

\[
R_j T_j = T_j R_j, \quad R_j R_i = R_i R_j
\]

(2.4)

\[
R_j^2 + T_j^2 = |z_j|^2 \Delta_j
\]

(2.5)

\[
\Delta_j R_i = (2 \delta_{ij} + R_j) \Delta_j
\]

for each \( i, j \) (of course, when applied to sufficiently smooth functions) where \( \delta_{ij} \) is the Kronecker delta. Note that each \( R_j \) preserves \( n \)-harmonicity by (2.5). We also remark that each \( R_j \) has the invariance property

\[
R_j (u \circ L) = (R_j u) \circ L
\]

(2.6)

for real linear transformations \( L \) on \( C^n \) of the form \( L(z) = (L_1(z_1), \ldots, L_n(z_n)) \) where \( L_i \) is a real linear transformation on \( C \).
In our argument below it seems more convenient to use real notation for differential operators. So, we let
\[ D_{x_j} = \partial_j + \overline{\partial}_j, \quad D_{y_j} = \sqrt{-1}(\partial_j - \overline{\partial}_j) \]
for \( j = 1, \ldots, n \). In terms of these real differential operators, note that
\[ \mathcal{R}_j = \Re z_j D_{x_j} + \Im z_j D_{y_j}, \quad T_j = \Re z_j D_{y_j} - \Im z_j D_{x_j} \]
for each \( j \). Here, \( \Re a \) and \( \Im a \) denote the real and imaginary part of \( a \in \mathbb{C} \), respectively.

By a multi-index we mean an \( n \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of nonnegative integers. We use conventional multi-index notation. Given a multi-index \( \alpha \), we let \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( \alpha! = \alpha_1! \cdots \alpha_n! \). Also, \( X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \) for \( X = (X_1, \ldots, X_n) \). We let
\[ D_X = (D_{x_1}, \ldots, D_{x_n}), \quad D_Y = (D_{y_1}, \ldots, D_{y_n}) \]
and
\[ \mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n), \quad T = (T_1, \ldots, T_n) \]
for the purpose of multi-index notation.

By a componentwise rotation \( \rho \) on \( \mathbb{C}^n \), we mean that \( \rho \) is a rotation of the form \( \rho(z) = (\zeta_1 z_1, \ldots, \zeta_n z_n) \), \( \zeta \in \mathbb{T}^n \). Note that each \( \mathcal{R}_j \) is componentwise rotation-invariant by (2.6). The same invariance property for each \( T_j \) is also easily verified. As a consequence, we have componentwise rotation-invariance property for general mixed differential operators:
\[ \mathcal{R}^\alpha T^\beta (u \circ \rho) = (\mathcal{R}^\alpha T^\beta u) \circ \rho \]
for componentwise rotations \( \rho \) and multi-indices \( \alpha, \beta \). This property will be useful in normalizing our argument later.

Finally, we use the notation \( \Lambda \) for the set of all multi-indices \( \alpha \) such that each component of \( \alpha \) is 0 or 1 and put
\[ e = (1, 1, \ldots, 1) \in \mathbb{T}^n \]
for simplicity.

**Proposition 2.2.** Suppose that \( u \in \mathcal{H}^\infty \) is continuous on \( D^n \cup W \) for some boundary open set \( W \). If both \( u \) and \( \mathcal{R}^\alpha u \) vanish on \( W \) for all \( \alpha \in \Lambda \), then \( u = 0 \) on \( D^n \).

**Proof.** Since \( u = 0 \) on \( W \), \( u \) extends to an \( n \)-harmonic function across \( W \), as mentioned above. We claim
\[ \mathcal{R}^\alpha u = 0 \quad \text{on} \quad W \]
for all multi-indices \( \alpha \). In order to see this, we introduce temporary notation. Given an integer \( m \geq 1 \), let \( \Lambda_m \) be the set of all multi-indices \( \alpha \) such that \( \alpha_i \leq m \) for each \( i \). Note that we have (2.8) for \( \alpha \in \Lambda = \Lambda_1 \) by assumption. We now proceed by induction on \( m \). Suppose that (2.8) holds for all \( \alpha \in \Lambda_m \). Let \( \alpha \in \Lambda_m \) and fix \( j \). If \( \alpha_j \leq m - 1 \), then \( \mathcal{R}_j \mathcal{R}^\alpha u \) vanishes on \( W \) by induction hypothesis. Consider the case \( \alpha_j = m \). Put \( f = \mathcal{R}^{\alpha_1}_1 \cdots \mathcal{R}^{\alpha_{j-1}}_{j-1} \mathcal{R}^{m-1}_{j} \mathcal{R}^{\alpha_{j+1}}_{j+1} \cdots \mathcal{R}^\alpha_n u \) for simplicity. Note that \( f \) is \( n \)-harmonic across \( W \), because \( u \) is. Thus, we have by (2.4)
\[ \mathcal{R}_j \mathcal{R}^\alpha u = \mathcal{R}^\alpha_j f = -T^2_j f \quad \text{on} \quad W. \]
Moreover, \( T^2_j f = 0 \) on \( W \), because \( f = 0 \) on \( W \) by induction hypothesis. So, we see that (2.8) holds for all \( \alpha \in \Lambda_{m+1} \) and the induction is complete.

Now, let \( \zeta \in W \) be an arbitrary but fixed point. Then it follows from (2.8) that \( T^\beta \mathcal{R}^\alpha u(\zeta) = 0 \) for all multi-indices \( \alpha \) and \( \beta \). Note that we may assume \( \zeta = e \) by using componentwise rotation-invariance (2.7). So, \( T^\beta \mathcal{R}^\alpha u(e) = 0 \) for all \( \alpha \) and \( \beta \). On
the other hand, we see by routine and straightforward calculations that $T^β R^α u(e)$ is of the form

$$
(2.10) \quad T^β R^α u(e) = D^α u(e) + \sum_{\text{lower order}} c_{α', β'} D^α u(e);
$$

the sum is to be taken over all multi-indices $α'$ and $β'$ with $|α'| < |α|$ or $|β'| < |β|$. Thus, by an inductive argument, we obtain $D^α D^β u(e) = 0$ for all multi-indices $α$ and $β$. So, by real-analyticity, we conclude $u = 0$ in some open subset of $D^n$ and thus on the whole $D^n$. The proof is complete.

**Remark.** (1) Compared with the ball version (Proposition 4.1 of [7]) of Proposition 2.2, the hypothesis that “$R^α u = 0$ for all $α \in Λ$” might seem too much at a glance. In the same context, one might expect that “$R_j u = 0$ for all $j$” would be enough. It turns out that such hypothesis requiring all $α \in Λ$ is essential, which shows a quite different nature caused by the fact that the polydisk is a product domain. To see an example, let $g$ be the harmonic extension on $D$ of a nonzero function continuous on $T$ and vanishing on some open subarc $I$. Define $u(z) = g(z_1) \cdots g(z_n)$ for $z \in D^n$ and put $W = I^n$. Clearly, we have $u \neq 0$, $u \in h^∞ \cap C(D^n)$ and $u = 0$ on $W$. Moreover, $R^α u = 0$ on $W$ for all $α \in Λ$ with $α \neq (1, \ldots, 1)$. Thus, just one single missing $α \in Λ$ does not guarantee the triviality of $u$.

(2) In the proof of Proposition 2.2, note that the boundedness of given function is used only to ensure the smooth extension across $W$ in each variable separately. Consider a function $u n$-harmonic on $D^n$ and continuous on $U \cap D^n$ for some open set $U$ with $W := U \cap T^n \neq 0$. If $u = 0$ on $W$, then one can use the reflection principle to see that $u$ extends $n$-harmonically across $W$. So, Proposition 2.2 remains valid for such a function $u$.

3. ZERO PRODUCTS

In this section we prove our theorems. We need several technical lemmas. First, we begin with an observation on how Taylor approximations near distinguished boundary points behave when a bounded $n$-harmonic function vanishes on some boundary open set.

Given an integer $0 \leq k \leq n$, we let $Λ(k)$ be the set of all $α \in Λ$ such that $|α| = k$. Also, for a multi-index $β$, we say $β \in Λ^*(k)$ if there exists $α \in Λ(k)$ such that $β_i = α_i + 1$ for some $i$ and $β_j = α_j$ for all $j \neq i$.

We let

$$
τ(z) = (1 - ℜz_1, \ldots, 1 - ℜz_n)
$$

$$
η(z) = (|1 - z_1|, \ldots, |1 - z_n|)
$$

for $z \in C^n$.

**Lemma 3.1.** Let $u \in h^∞ \cap C(D^n \cup W)$ for some boundary open set $W$. Suppose that $u$ is not identically 0 on $D^n$ and vanishes on $W$. Then there exist some $ζ \in W$ and an integer $1 \leq k \leq n$ such that

$$
u(ζ) = \sum_{α \in Λ(k)} (-1)^{|α|} R^α u(ζ) τ(ζ)^α + O \left( \sum_{β \in Λ^*(k)} η(ζ)^β \right)
$$

for $ζ \in D^n \cup W$ near $e$ where $ζ = (ζ_1 z_1, \ldots, ζ_n z_n)$. Moreover, some coefficient $R^α u(ζ) \neq 0$. 

Proof. Since \( u = 0 \) on \( W \), \( u \) extends to an \( \nu \)-harmonic function across \( W \), as mentioned earlier. Since \( |u|_W = 0 \) and \( u \not\equiv 0 \) on \( D^n \) by assumption, we have \( R^\alpha u|_W \not\equiv 0 \) for some \( \alpha \in \Lambda \) by Proposition 2.2. So, the following minimum is well defined and positive:

\[
(3.1) \quad k = \min \{|\alpha| : R^\alpha u|_W \not\equiv 0, \alpha \in \Lambda\}.
\]

Now, choose a point \( \zeta \in W \) such that \( R^\alpha u(\zeta) \not\equiv 0 \) for some \( \alpha \in \Lambda \) that achieves the above minimum. Note that the function \( v(z) := u(\zeta z) \) is real-analytic near \( e \). Let

\[
(3.2) \quad v(z) = \sum_{\alpha, \beta} (-1)^{|\alpha|} \frac{D^\alpha D^\beta v(e)}{\alpha! \beta!} \tau(z)^\alpha Y^\beta
\]

be the Taylor series expansion of \( v \) at \( e \). Here, \( Y = (\Im z_1, \ldots, \Im z_n) \). For the coefficients above, note that \( D^\alpha v(e) = R^\alpha v(e) = R^\alpha u(\zeta) \) for \( \alpha \in \Lambda(k) \) by (2.7). Thus, we can rewrite the sum in (3.2) as

\[
(3.3) \quad v(z) = \sum_{\alpha \in \Lambda(k)} (-1)^{|\alpha|} R^\alpha u(\zeta) \tau(z)^\alpha + \text{Error}.
\]

We now estimate the error term. Consider an arbitrary multi-index \( \alpha \) with \( N(\alpha) \leq k - 1 \) where \( N(\alpha) \) denotes the number of nonzero components of \( \alpha \). Associated with such an \( \alpha \) is a multi-index \( \beta = (\beta_1, \ldots, \beta_n) \) where \( \beta_j = 0 \) if \( \alpha_j = 0 \) and \( \beta_j = 1 \) otherwise. Then, since \( \beta \in \Lambda \) and \( |\beta| = N(\alpha) \leq k - 1 \), we have \( R^\beta u|_W = 0 \) by minimality of \( k \). Suppose that there is some \( j \) such that \( \alpha_j > \beta_j = 1 \). Let \( \tilde{\beta} = (\beta_1, \ldots, \beta_{j-1}, 0, \beta_{j+1}, \ldots, \beta_n) \) and put \( f = R^\beta u \). Then we also have \( R^\beta u|_W = 0 \) by minimality of \( k \). So, we have

\[
R_j R^\beta u = R_j^2 f = -T^2 f = 0 \quad \text{on} \quad W
\]

by the same argument as in (2.9). Now, repeating the same reasoning as many times as needed, we obtain \( R^\alpha u|_W = 0 \). In addition, we claim

\[
(3.4) \quad R^\alpha T^\beta u(\zeta) = 0
\]

for all multi-indices \( \beta \). To show this, we assume \( \alpha = (\alpha_1, \ldots, \alpha_{k-1}, 0, \ldots, 0) \) for notational simplicity. Also, shrinking \( W \) if necessary, we may assume \( W = W_1 \times \cdots \times W_n \) where each \( W_j \) is an open arc in \( T \) containing \( \zeta_j \). Then it follows from Proposition 2.2 that each function \( u(\cdot, \zeta) \), \( \zeta \in W_k \times \cdots \times W_n \), is identically zero on \( D^{k-1} \). Therefore, given a multi-index \( \beta \), we have

\[
R^\alpha T^{\beta_1} \cdots T_{k-1}^{\beta_{k-1}} u = 0 \quad \text{on} \quad D^{k-1} \times W_k \times \cdots \times W_n
\]

and therefore

\[
T_k^{\beta_k} \cdots T_n^{\beta_n} R^\alpha T^{\beta_1} \cdots T_{k-1}^{\beta_{k-1}} u = 0 \quad \text{on} \quad D^{k-1} \times W_k \times \cdots \times W_n.
\]

This yields (3.4) as a special case, because \( R_i T_j = T_j R_i \) for each \( i, j \).

Note that we have \( R^\alpha T^\beta v(e) = 0 \) by (3.4) and (2.7). So, by (2.10) and (3.4), we have \( D^\alpha D^\beta v(e) = 0 \) for all \( \alpha \) with \( N(\alpha) \leq k - 1 \) and for all \( \beta \). Note that \( |\alpha| = N(\alpha) = k \) if and only if \( \alpha \in \Lambda(k) \). Therefore, the error term in (3.3) can be decomposed into three pieces

\[
\sum_{|\alpha| > N(\alpha) = k} + \sum_{|\alpha| = k, |\beta| \geq 1} + \sum_{|\alpha| > K, |\beta| \geq 0} (-1)^{|\alpha|} \frac{D^\alpha D^\beta v(e)}{\alpha! \beta!} \tau(z)^\alpha Y^\beta.
\]

Note that \( |\tau(z)^\alpha Y^\beta| \leq \eta(z)^{\alpha+\beta} \) for all \( \alpha \) and \( \beta \). Hence, it is easily seen that each sum above is dominated by \( \sum_{\gamma \in \Lambda^*} \eta(z)^\gamma \), as desired. The proof is complete.\( \square \)
Remark. The proof above shows that the point \( \zeta = \zeta_u \in W \) can be chosen arbitrarily, once \( R^\alpha u(\zeta) \neq 0 \) for some \( \alpha \in \Lambda \) that achieves the minimum (3.1). Thus, given a finite number of functions \( u_1, \ldots, u_N \in L^\infty \cap C(D^n \cup W) \) which are not identically 0 on \( D^n \) and vanishes on some boundary open set \( W \), one may easily modify the proof above to choose points \( \zeta_{u_i} \in W \) so that \( \zeta_{u_1} = \cdots = \zeta_{u_N} \). This fact will be used later in the proof of Theorem 1.2 and Theorem 1.3.

Before proving our main results, we need some technical estimates which are essentially proved in [7].

First, we need an integral estimate. Given \( c, s \) and \( d \) real, we let
\[
\Phi_{c,s}(a,b) = 1 + \left\| \log(1 - |a|)(1 - |b|) \right\|^s \frac{|1 - a/b|^2 + c}{1 - a/b}.
\]
and consider corresponding integral
\[
I_{c,s,d}(a,b) = \int_D \frac{|1 - \xi|^d \Phi_{c,s}(\xi,b)}{|1 - a\xi|^2} dV_1(\xi)
\]
for \( a, b \in D \).

**Lemma 3.2.** Given \( d, s \geq 0 \) and \( c \geq d - 2 \), there is a constant \( C = C(c, s, d) \) such that
\[
I_{c,s,d}(a,t) \leq C \times \begin{cases} (1 - t)^d \Phi_{0,s+1}(a,t) & \text{if } c > d \\ \Phi_{c-d,s+1}(a,t) & \text{if } c \leq d \end{cases}
\]
for \( a \in D \) and \( 0 < t < 1 \).

**Proof.** This follows from Proposition 3.8 of [7]. \(\square\)

Next, we need a couple of estimates showing how a certain 1-dimensional Toeplitz operator behaves against test functions. Let \( S_\sigma \) be the 1-dimensional Toeplitz operator with symbol \( \sigma \) where
\[
\sigma(a) = 1 - \Re a
\]
for \( a \in D \), or more explicitly,
\[
S_\sigma f(a) = \int_D \frac{f(\xi)(1 - \Re \xi)}{(1 - a\xi)^2} dV_1(\xi), \quad a \in D
\]
for \( f \in A^2(D) \). Also, we let
\[
\mu_t(a) = \frac{1}{1 - ta}, \quad a \in D
\]
for each \( 0 < t < 1 \).

**Lemma 3.3.** Let \( m > 2 \) be an integer. Then there exists a polynomial \( \mathcal{P} \) in two variables with no term of degree less than 2 such that
\[
S_\sigma \mu_t^m = \frac{1}{m-1} \mu_t^{m-1} + \mu_t^m \mathcal{P}(1 - t, \mu_t^{-1}) + O(\mu_t^s)
\]
for \( \frac{1}{2} < t < 1 \).

**Proof.** See the proof of Lemma 4.3 in [7]. \(\square\)
Lemma 3.4. Let \( \ell \) and \( m \) be positive integers such that \( m > \ell + 1 \). Then
\[
S_\ell \mu_t^m = \frac{\mu_t^{m-\ell}}{(m-1) \cdots (m-\ell)} \{ 1 + E_\ell \}
\]
for \( \frac{1}{2} < t < 1 \) where \( \{ E_\ell \} \) are some uniformly bounded functions such that \( E_\ell(t) = o(1) \) as \( t \to 1 \).

Proof. See the proof of Theorem 1.2 in [7].

We are now ready to prove our theorems. As is seen in the proofs of Proposition 2.2 and Lemma 3.1, the componentwise rotation-invariance property (2.7) allows us to normalize our argument. We will do the same normalization in the proofs below. This time, however, such normalization effects not only symbol functions but also Toeplitz operators with those symbols. So, we need to make that point clear. Let \( \rho \) be a componentwise rotation and let \( C_\rho \) denote the composition operator \( f \mapsto f \circ \rho \) on \( A^2 \). Then it is straightforward to verify (for \( N = 1 \) and thus for general \( N \)) \[
C_\rho(T_{u_1} \cdots T_{u_N})C_{\rho^{-1}} = T_{u_1 \circ \rho} \cdots T_{u_N \circ \rho},
\]
which in turn yields
\[
T_{u_1} \cdots T_{u_N} = 0 \iff T_{u_1 \circ \rho} \cdots T_{u_N \circ \rho} = 0
\]
for any finite number of Toeplitz operators \( T_{u_1}, \ldots, T_{u_N} \).

Given \( t = (t_1, \ldots, t_n) \) where \( 0 < t_j < 1 \), we let \[
\lambda_t(z) = \mu_{t_1}(z_1) \cdots \mu_{t_n}(z_n) \quad z \in D^n;
\]
this will be the source for our test functions.

Note on Constants. For two positive quantities \( X \) and \( Y \), we often write \( X \lesssim Y \) or \( Y \gtrsim X \) if \( X \) is dominated by \( Y \) times some inessential positive constant. Also, we write \( X \asymp Y \) if \( X \lesssim Y \leq X \).

We first prove Theorem 1.1.

Proof of Theorem 1.1. Assume \( T_{u_1} T_{u_2} = 0 \). Then, since \( u_1 \) and \( u_2 \) are both continuous on \( D \cup W \) by assumption, we have \[
0 = (T_{u_1} T_{u_2})^* = u_1 u_2 \quad \text{on} \quad W
\]
by Proposition 2.1. There are two cases to consider: (i) Both \( u_1 \) and \( u_2 \) vanish on \( W \) (ii) Either \( u_1 \) or \( u_2 \) does not vanish on some boundary open subset of \( W \). In case (i) we have by Lemma 3.1 \( u_1, u_2 \in \text{Li}[p_1(U \cap \overline{D^n})] \) for some open set \( U \) with \( U \cap T^n \neq \emptyset \). Thus, the case (i) is contained in Theorem 1.3 to be proved below.

So, we may assume (ii). Note that we may further assume that \( u_1 \) does not vanish on some boundary open set, still denoted by \( W \), because otherwise we can use the adjoint operator \( (T_{u_1} T_{u_2})^* = T_{u_2}^* T_{u_1} \). We now have \( u_2 = 0 \) on \( W \). Assume that \( u_2 \) is not identically 0 on \( D^n \). This will lead us to a contradiction.

In the rest of the proof, we let \( z = (z_1, \ldots, z_n) \in D^n \) represent an arbitrary point. Since \( u_2 = 0 \) on \( W \) and \( u_2 \) is not identically 0 on \( D^n \), we have a point \( \zeta \in W \) and an integer \( k \geq 1 \) provided by Lemma 3.1 (with \( u_2 \) in place of \( u \)). By componentwise
rotation-invariance (2.7) and (3.5), we may assume \( \zeta = e \) without loss of generality. Also, put \( C_\alpha = R^\alpha u_2(e) \) for multi-indices \( \alpha \). Then, by Lemma 3.1, we have

\[
u_2(z) = \sum_{\alpha \in \Lambda(k)} C_\alpha \tau(z)^\alpha + \mathcal{O} \left( \sum_{\beta \in \Lambda^+(k)} \eta(z)^\beta \right)
\]

for \( z \in D^0 \cup W \) near \( e \) where some coefficient \( C_\alpha \neq 0 \). Put \( c_1 = u_1(e) \neq 0 \) and

\[
m_2(z) = \sum_{\alpha \in \Lambda(k)} C_\alpha \tau(z)^\alpha
\]

for simplicity. Also, let \( e_1 = u_1 - c_1 \) and \( e_2 = u_2 - m_2 \). Then we have

\[0 = T_{u_1} T_{u_2} = T_{e_1} T_{m_2} = c_1 T_{m_2} + T_{e_1} T_{m_2} + T_{u_1} T_{e_2}\]

and thus

\[-c_1 T_{m_2} = T_{e_1} T_{m_2} + T_{u_1} T_{e_2}.
\]

Now we apply each side of the above to the same test functions and derive a contradiction. Here, we will use test functions \( \lambda_{t}^m \) where \( m > 4 \) is any fixed integer and \( t = (t_1, \ldots, t_n) \) is chosen as follows.

**Choice of t**: Put

\[F(x) = \sum_{\alpha \in \Lambda(k)} C_\alpha x^\alpha \]

for \( x \in \mathbb{R}^n \). Note that \( F \) is a non-zero polynomial on \( \mathbb{R}^n \), because some coefficient \( C_\alpha \) is not zero. Thus, there is some \( y \in (0, 1)^n \) such that \( F(y) \neq 0 \). Given \( t \in (0, 1) \), we now choose \( t_j = t_j(t) \) such that

\[t_j = \sqrt{1 - y_j(1 - t_j^2)} \quad j = 1, \ldots, n,
\]

and let \( t = t(t) = (t_1, \ldots, t_n) \) for the rest of the proof. Note \( t_j \in (0, 1) \) and \( 1 - t_j^2 = y_j(1 - t_j^2) \) for each \( j \) so that

\[
\prod_{j=1}^{n} (1 - t_j^2)^{\alpha_j} = (1 - t_j^2)^k y^\alpha
\]

for each \( \alpha \in \Lambda(k) \).

**Estimate of \( T_{m_2} \lambda_{t}^m(t) \)**: Note that \( \tau(z) = (\sigma(z_1), \ldots, \sigma(z_n)) \). Thus, we have

\[
T_{m_2} \lambda_{t}^m(z) = \sum_{\alpha \in \Lambda(k)} C_\alpha \prod_{j=1}^{n} \int_D \frac{\sigma^\alpha_j(w_j) t^m_{\sigma_j}(w_j)}{(1 - z_j w_j)^2} dV_1(w_j)
\]

(3.7)

\[
= \sum_{\alpha \in \Lambda(k)} C_\alpha \prod_{j=1}^{n} S^\alpha_{\sigma_j} t^m_{\sigma_j}(z_j)
\]

so that

\[
T_{m_2} \lambda_{t}^m(t) = \sum_{\alpha \in \Lambda(k)} C_\alpha \left( \prod_{j=1}^{n} S_{\sigma_j} t^m_{\sigma_j}(t_j) \right) \left( \prod_{\alpha_j=0}^{m} t^\alpha_j(t_j) \right).
\]

Note that, by Lemma 3.3, we have

\[
S_{\sigma_j} t^m_{\sigma_j}(a) = \frac{\mu^{m-1}_{\sigma_j}(a)}{m-1} \{ 1 + \mathcal{O}([1 - ta]) \}
\]

(3.8)
as $t \to 1$ uniformly in $\alpha \in D$ and hence
\[
S_{\sigma} \mu_{\delta}^n(t) = \frac{\mu_{\delta}^{m-1}(t)}{m-1} (1 + O(1 - t))
\]
as $t \to 1$. It follows that
\[
T_{m_{\sigma}} \lambda_{m_{\sigma}}(t) = \frac{1 + o(1)}{(m - 1)^k} \sum_{\alpha \in \Lambda(k)} C_{\alpha} \left( \prod_{\alpha_j = 1} \mu_{\delta_j}^{m-1}(t_j) \right) \left( \prod_{\alpha_j = 0} \mu_{\delta_j}^{m}(t_j) \right)
\]
\[
= \frac{\lambda_{m_{\sigma}}(1 + o(1))}{(m - 1)^k} \sum_{\alpha \in \Lambda(k)} C_{\alpha} \left( \prod_{\alpha_j = 1} (1 - t_j^2) \right)
\]
\[
= \frac{\lambda_{m_{\sigma}}(1 + o(1))}{(m - 1)^k} (1 - t^2)^k F(y) \quad \text{by (3.6)}
\]
and thus
\[
|T_{m_{\sigma}} \lambda_{m_{\sigma}}(t)| \gtrsim \lambda_{m_{\sigma}}(1 - t^2)^k |F(y)| = \frac{|F(y)|}{(1 - t^2)^{mn-k}}
\]
as $t \to 1$. Since $F(y) \neq 0$, we finally get
\[
(3.9) \quad |T_{m_{\sigma}} \lambda_{m_{\sigma}}(t)| \gtrsim \frac{1}{(1 - t^{2})^{mn-k}}
\]
as $t \to 1$.

**Estimate of $T_{u_{1}} T_{u_{2}} \lambda_{m_{\sigma}}(t)$:** First, recall that
\[
|e_{2}(z)| \lesssim \sum_{\beta \in \Lambda^{*}(k)} |1 - z_{1}|^{\beta_{1}} \cdots |1 - z_{n}|^{\beta_{n}}.
\]
Also, for $\beta \in \Lambda^{*}(k)$, note $\beta_{j} \leq 2 < m - 2$ for each $j$ and $|\beta| = k + 1$. Thus, by Lemma 3.2, we have
\[
|T_{u_{2}} \lambda_{m_{\sigma}}(z)| \lesssim \sum_{\beta \in \Lambda^{*}(k)} \prod_{j=1}^{n} \int_{D} \frac{|1 - u_{j}|^{\beta_{j}}}{|1 - z_{j} u_{j}|^{2} |1 - t_{j} u_{j}|^{m}} dV_{1}(u)
\]
\[
\lesssim \sum_{\beta \in \Lambda^{*}(k)} \prod_{j=1}^{n} \Phi_{0,1}(z_{j}, t_{j}) (1 - t_{j})^{m-2-\beta}
\]
\[
\lesssim \frac{1}{(1 - t)^{mn-2n-k-1}} \prod_{j=1}^{n} \Phi_{0,1}(z_{j}, t_{j})
\]
and hence, again by Lemma 3.2, we obtain
\[
|T_{u_{1}} T_{u_{2}} \lambda_{m_{\sigma}}(t)| \lesssim \frac{1}{(1 - t)^{nm-2n-k-1}} \prod_{j=1}^{n} \int_{D} \frac{|\Phi_{0,1}(u_{j}, t_{j})|}{|1 - t_{j} u_{j}|^{2}} dV_{1}(u_{j})
\]
\[
\lesssim \frac{1}{(1 - t)^{nm-2n-k-1}} \prod_{j=1}^{n} \Phi_{0,2}(t_{j}, t_{j})
\]
as $t \to 1$. Note that
\[
\prod_{j=1}^{n} \Phi_{0,2}(t_{j}, t_{j}) = \prod_{j=1}^{n} \frac{1 + |\log(1 - t_{j})|^{2}}{(1 - t_{j}^{2})^{2}} \approx \frac{1 + |\log(1 - t)|^{2n}}{(1 - t)^{2n}}
\]
as \( t \to 1 \). Combining these estimates, we conclude

\[
|T_{u_1} T_{e_2} \Lambda^m_t(t)| \lesssim \frac{1 + |\log(1-t)|^{2n}}{(1-t)^{nm-k-1}} = o(1)
\]
as \( t \to 1 \).

**Estimate of** \( T_{e_1} T_{m_2} \Lambda^m_t(t) \): By (3.7), we first note

\[
T_{m_2} \Lambda^m_t(z) = \sum_{\alpha \in \Lambda(k)} C_\alpha \left( \prod_{\alpha_j=1} S_{\sigma_j} \mu_{t_j}^{m_j}(z_j) \right) \left( \prod_{\alpha_j=0} \mu_{t_j}^{m_j}(z_j) \right)
\]

and hence by (3.8)

\[
|T_{m_2} \Lambda^m_t(z)| \lesssim \sum_{\alpha \in \Lambda(k)} |C_\alpha| \left( \prod_{\alpha_j=1} |\mu_{t_j}^{m_j-1}(z_j)| O(1) \right) \left( \prod_{\alpha_j=0} |\mu_{t_j}^{m_j}(z_j)| \right)
\]

\[
\lesssim |\lambda^m_t(z)| \sum_{\alpha \in \Lambda(k)} |C_\alpha| \left( \prod_{\alpha_j=1} |1 - t_j z_j| \right)
\]

\[
= |\lambda^m_t(z)| \sum_{\alpha \in \Lambda(k)} |C_\alpha| |1 - t_1 z_1|^{\alpha_1} \cdots |1 - t_n z_n|^{\alpha_n}
\]

for \( z \in D^n \). Since coefficients \( C_\alpha \) are bounded, it follows that

\[
|T_{e_1} T_{m_2} \Lambda^m_t(t)| \lesssim \sum_{\alpha \in \Lambda(k)} \int_{D^n} \frac{|e_1(w)|}{|1 - t_j w_j|^{m+2-\alpha_j}} dV(w).
\]

(3.11)

Given \( \epsilon > 0 \) small, we let

\[
D_j(\epsilon) = \{ z_j \in D : |z_j - 1| < \epsilon \}
\]

for \( j = 1, \ldots, n \) and put

\[
\Omega_\epsilon = D_1(\epsilon) \times D_2(\epsilon) \times \cdots \times D_n(\epsilon).
\]

Corresponding to this region, we now decompose the sum in (3.11) into two pieces as follows:

\[
\sum_1 = \sum_{\alpha \in \Lambda(k)} \int_{D^n \setminus \Omega_\epsilon} \quad \text{and} \quad \sum_2 = \sum_{\alpha \in \Lambda(k)} \int_{\Omega_\epsilon}
\]

for convenience.

We first estimate \( \sum_1 \). Note that \( D^n \setminus \Omega_\epsilon \) is the finite union of sets of the form \( J = J_1 \times J_2 \times \cdots \times J_n \) where each \( J_j \) is either \( D_j(\epsilon) \) or \( D \setminus D_j(\epsilon) \) and at least one \( J_j \) is \( D \setminus D_j(\epsilon) \). Given such \( J \), let \( \ell = \ell(J) \) be an index such that \( J_\ell = D \setminus D_\ell(\epsilon) \). Since

\[
|1 - t_\ell w_\ell| \geq |1 - w_\ell| - (1 - t_\ell) \geq \epsilon - \epsilon(1 - t) \geq \epsilon - y(1 - t)
\]

for \( w_\ell \in J_\ell \) as \( t \to 1 \), we have, for \( \alpha \in \Lambda(k), \)

\[
\int_{J_\ell} \frac{dV_1(w_\ell)}{|1 - t_\ell w_\ell|^{m+2-\alpha_\ell}} \lesssim \frac{1}{\epsilon - |y(1 - t)|^{m+2}}
\]

for convenience.
as $t \to 1$. On the other hand, we have
\begin{equation}
\int_D \frac{dV_1(w)}{[1 - t_j w_j]^{m + 2 - \alpha}} \approx \frac{1}{(1 - t_j)^{m - \alpha}}, \quad j \neq \ell;
\end{equation}
see Lemma 4.2.2 of [15] for a proof of this well-known estimate. Note that
\[
\prod_{j \neq \ell} \frac{1}{(1 - t_j)^{m - \alpha_j}} \approx \prod_{j \neq \ell} \frac{1}{(1 - t)^{m - \alpha_j}}
= \frac{(1 - t)^{-\alpha}}{(1 - t)^{m - \alpha}}
\leq \frac{(1 - t)^{-1}}{(1 - t)^{m - k}}
\]
because $|\alpha| = k$. It follows that
\[
\int \prod_{j=1}^n \frac{1}{[1 - t_j w_j]^{m + 2 - \alpha_j}} dV(w) \lesssim \frac{1}{(1 - t)^{mn - k}} \cdot (1 - t)^{m - 1} \cdot |\nu - y(1 - t)|^{m + 2}
\]
as $t \to 0$. Note that this estimate is uniform in $J$ and $\alpha \in \Lambda(k)$ as $t \to 1$. So, since $\epsilon_1$ is bounded, we conclude
\begin{equation}
\sum_1 \lesssim \frac{\nu(\epsilon, t)}{(1 - t)^{mn - k}}
\end{equation}
as $t \to 1$ where $\nu(\epsilon, t) = (1 - t)^{m - 1} / (\epsilon - |\nu - y(1 - t)|^{m + 2}$.

Next, we estimate $\sum_2$. Note that $|w - \epsilon| < \sqrt{n} \epsilon$ for $w \in \Omega_\epsilon$. Thus, setting
\[
\omega(\epsilon) = \sup \{|u_1(w) - u_1(\epsilon)| : w \in D^n \cup W, \quad |\epsilon - w| < \sqrt{n} \epsilon\},
\]
we have
\begin{equation}
\sum_2 \lesssim \omega(\epsilon) \sum_{\alpha \in \Lambda(k)} \int_D \prod_{j=1}^n \frac{1}{[1 - t_j w_j]^{m + 2 - \alpha_j}} dV(w)
\lesssim \omega(\epsilon) \sum_{\alpha \in \Lambda(k)} \prod_{j=1}^n \frac{1}{(1 - t_j)^{m - \alpha_j}}
\approx \frac{\omega(\epsilon)}{(1 - t)^{mn - k}}
\end{equation}
where the second inequality holds by (3.12).

In summary, with $\epsilon > 0$ small and fixed, we have by (3.13) and (3.14)
\begin{equation}
|T_{c_1} T_{m_2} \Lambda_1^n(t)| \lesssim \omega(\epsilon) + \nu(\epsilon, t)
\end{equation}
as $t \to 1$ and the estimate is uniform in $\epsilon$.

**Finish of Proof**: Setting $M = -c_1 T_{m_2}$ and $R = T_{c_1} T_{m_2} + T_{u_1} T_{c_2}$, we obtain from (3.9), (3.10) and (3.15) that
\[
1 = \frac{|R \Lambda_1^n(t)|}{|M \Lambda_1^n(t)|} \lesssim \omega(\epsilon) + \nu(\epsilon, t) + o(1), \quad \epsilon > 0 : \text{fixed}
\]
as $t \to 1$ and the estimate is uniform in $\epsilon$. So, now first taking the limit $t \to 1$ and then $\epsilon \to 0$, we have
\[
1 \lesssim \omega(\epsilon) \to 0
\]
Next, we prove Theorem 1.2.

Proof of Theorem 1.2. Assume $T_{u_1}T_{u_2}T_{u_3}T_{u_4} = 0$. Then, since each $u_j$ is continuous up to the distinguished boundary by assumption, we have

$$\left(T_{u_1}T_{u_2}T_{u_3}T_{u_4}\right) = u_1u_2u_3u_4 = 0 \quad \text{on} \quad T^n$$

by Proposition 2.1. Since $u_1u_2u_3u_4$ is continuous and vanishes everywhere on the distinguished boundary, there exists a boundary open set $W \subset T^n$ such that

either $u_j$ never vanishes on $W$,

or $u_j = 0$ on $W$

holds for each $j$. Since there is nothing to prove if $u_1 \equiv 0$, we may assume that $u_1|_W$ never vanishes.

Since each $u_j$ is not identically 0 on $D^n$, we may shrink (if necessary) the set $W$ to get a smaller boundary open set, still denoted by $W$, such that each $u_j$ satisfies

$$u_j(z) = \sum_{\alpha \in \Lambda(k_j)} R^\alpha u_j(\zeta) \tau(z)^\alpha + O \left( \sum_{\beta \in \Lambda^*(k_j)} \eta(z)^\beta \right)$$

for $z \in D^n \cup \overline{\zeta} W$ near $e$ where some coefficient $R^\alpha u_j(\zeta) \neq 0$. Note that such a point $\zeta$ can be chosen independently of $j$ by the remark mentioned right after Lemma 3.1. So, we may assume $\zeta = e$; this causes no loss of generality by (2.7) and (3.5). Also, we let

$$C_{j,\alpha} = R^\alpha u_j(e)$$

for each $j$ and multi-index $\alpha$. Using such notation, we rewrite (3.17) as

$$u_j(z) = \sum_{\alpha \in \Lambda(k_j)} C_{j,\alpha} \tau(z)^\alpha + O \left( \sum_{\beta \in \Lambda^*(k_j)} \eta(z)^\beta \right)$$

for $z \in D^n \cup W$ near $e$ where some coefficient $C_{j,\alpha} \neq 0$.

We let $k_j = 0$ if $u_j(e) \neq 0$. Note $k_j = 0$, because $u_1$ never vanishes on $W$. With such convention we have

$$|u_j(z)| \lesssim \sum_{\alpha \in \Lambda(k_j)} |1 - z_1|^\alpha \cdots |1 - z_n|^\alpha$$

for each $j$, because $\Lambda(0)$ consists of the zero multi-index in case $k_j = 0$. Here, and in the rest of the proof, $z \in D^n$ represents an arbitrary point. Now, we define the major part of $u_j$ by

$$m_j(z) := \sum_{\alpha \in \Lambda(k_j)} C_{j,\alpha} \tau(z)^\alpha$$
for each $j$. More explicitly, we define

$$m_j(z) = \begin{cases} u_j(e) & \text{if } u_j(e) \neq 0 \\ \sum_{\alpha \in \Lambda(k_j)} C_{j,\alpha} \tau(z)^\alpha & \text{if } u_j(e) = 0 \end{cases}$$

for each $j$. Put $e_j = u_j - m_j$ for each $j$. Note that $e_j(e) = 0$ for each $j$. Thus, we have

$$|e_j(z)| \leq \begin{cases} \sum_{i=1}^n |1 - z_i|^\epsilon & \text{if } u_j(e) \neq 0 \\ \sum_{\beta \in \Lambda^*(k_j)} |1 - z_1|^{\beta_1} \ldots |1 - z_n|^{\beta_n} & \text{if } u_j(e) = 0 \end{cases}$$

(3.20)

by the Lipschitz continuity hypothesis.

We introduce further notation. Put $M = T_{m_1} T_{m_2} T_{m_3} T_{m_4}$ and $R = T_{u_1} T_{u_2} T_{u_3} T_{u_4} - M$ for notational convenience. Then, setting

$$R_1 = T_{e_1} T_{m_2} T_{m_3} T_{m_4},$$
$$R_2 = T_{u_1} T_{e_2} T_{m_3} T_{m_4},$$
$$R_3 = T_{u_1} T_{u_2} T_{e_3} T_{m_4},$$
$$R_4 = T_{u_1} T_{u_2} T_{u_3} T_{e_4},$$

we have $R = R_1 + R_2 + R_3 + R_4$. Note that $R = -M$, because $T_{u_1} T_{u_2} T_{u_3} T_{u_4} = 0$ by assumption. Now we will estimate the same expression $M \lambda^m_3(t) = -R \lambda^m_3(t)$ in two different ways and reach a contradiction as in the proof of Theorem 1.1. Here, $m > 4$ is any fixed integer and $t = (t_1, \ldots, t_n)$ is chosen below. Put

$$k = k_1 + k_2 + k_3 + k_4.$$ 

Note that we have $k \geq 1$, because $(u_1 u_2 u_3 u_4)(e) = 0$. Also, recall that $k_1 = 0$.

**Choice of $t$:** First, we introduce some notation. Let $(\alpha, \beta, \gamma)$ denote an arbitrary triple of $\alpha \in \Lambda(k_2)$, $\beta \in \Lambda(k_3)$ and $\gamma \in \Lambda(k_4)$. Given $(\alpha, \beta, \gamma)$, we let

$$C_{\alpha + \beta + \gamma} = \prod_{\alpha_1 + \beta_1 + \gamma_1 \geq 1} \frac{1}{(m - 1) \ldots (m - \alpha_i - \beta_i - \gamma_i)}$$

for simplicity. Also, let $\tilde{\Lambda}$ be the set of all multi-indices of the form $\alpha + \beta + \gamma$ where $\alpha \in \Lambda(k_2)$, $\beta \in \Lambda(k_3)$ and $\gamma \in \Lambda(k_4)$. Using such notation, we consider two polynomials on $\mathbb{R}^n$ given by

$$G(x) : = \sum_{(\alpha, \beta, \gamma)} C_{2, \alpha} C_{3, \beta} C_{4, \gamma} x^{\alpha + \beta + \gamma}$$

$$= \sum_{\alpha + \beta + \gamma = h} \left( \sum_{\alpha + \beta + \gamma = h} C_{2, \alpha} C_{3, \beta} C_{4, \gamma} \right) x^h$$

(3.21)
and

\[ F(x) := \sum_{(\alpha, \beta, \gamma)} C_{2,\alpha} C_{3,\beta} C_{4,\gamma} C_{\alpha+\beta+\gamma} x^{\alpha+\beta+\gamma} \]

(3.22)

\[ = \sum_{h \in \Lambda} C_h \left( \sum_{\alpha+\beta+\gamma = h} C_{2,\alpha} C_{3,\beta} C_{4,\gamma} \right) x^h \]

for \( x \in \mathbb{R}^n \). It is clear from (3.21) and (3.22) that \( G \) is non-trivial if and only if \( F \) is.

Note that \( G \) is the product of non-trivial polynomials \( G_2, G_3 \) and \( G_4 \) defined by \( G_4(x) = \sum_{\omega \in \Lambda(k_4)} C_{j,\omega} x^\omega \) for \( j = 2, 3, 4 \). The polynomial \( G \) is therefore non-trivial. So, we conclude that the polynomial \( F \) is also non-trivial.

Now, as in the proof of Theorem 1.1, we choose \( y \in (0, 1)^n \) with \( F(y) \neq 0 \) and define

\[ t_i = t_i(t) = \sqrt{1 - y_i(1 - t^2)}, \quad 0 < t < 1, \]

for \( i = 1, \ldots, n \). Let \( t = (t_1, \ldots, t_n) \) for the rest of the proof. Note \( t_i \in (0, 1) \) and \( 1 - t_i^2 = y_i(1 - t^2) \) for each \( i \). Also, note that \( |\alpha + \beta + \gamma| = k_2 + k_3 + k_4 = k \) for all \( (\alpha, \beta, \gamma) \). So, we have

(3.23)

\[ \prod_{i=1}^n (1 - t_i^2)^{\alpha_i + \beta_i + \gamma_i} = (1 - t^2)^k y^{\alpha + \beta + \gamma} \]

for each \( (\alpha, \beta, \gamma) \).

**Estimate of \( M \lambda^m_k(t) \):** Note that \( M = c T_{m_2} T_{m_3} T_{m_4} \) where \( c = u_1(\omega) \neq 0 \). Using (3.19) and repeating arguments similar to (3.7), we have

\[ T_{m_2} T_{m_3} T_{m_4} \lambda^m_k(z) = \sum_{(\alpha, \beta, \gamma)} C_{2,\alpha} C_{3,\beta} C_{4,\gamma} \left( \prod_{i=1}^n S_{\sigma_i}^{\alpha+\beta+\gamma_i} \mu_i^m(t_i) \right). \]

Meanwhile, given \((\alpha, \beta, \gamma)\), we deduce from Lemma 3.4 that

\[ \prod_{i=1}^n S_{\sigma_i}^{\alpha+\beta+\gamma_i} \mu_i^m(t_i) \]

\[ = \left( \prod_{\alpha_i + \beta_i + \gamma_i = 0} \mu_i^m(t_i) \right) \left( \prod_{\alpha_i + \beta_i + \gamma_i \geq 1} S_{\sigma_i}^{\alpha+\beta+\gamma_i} \mu_i^m(t_i) \right) \]

\[ = \left( \prod_{\alpha_i + \beta_i + \gamma_i = 0} \mu_i^m(t_i) \right) \left( \prod_{\alpha_i + \beta_i + \gamma_i \geq 1} \mu_i^m \frac{1 + o(1)}{m - 1} \cdot \frac{1 + o(1)}{m - 2} \cdots \frac{1 + o(1)}{m - \alpha_i - \beta_i - \gamma_i} \right) \]

\[ = C_{\alpha+\beta+\gamma} \lambda^m_k(t) \{ 1 + o(1) \} \prod_{\alpha_i + \beta_i + \gamma_i \geq 1} \left( 1 - t_i^2 \right)^{\alpha_i + \beta_i + \gamma_i}, \]

as \( t \to 1 \) (hence all \( t_i \to 1 \)). Also, note that we have

\[ \prod_{\alpha_i + \beta_i + \gamma_i \geq 1} (1 - t_i^2)^{\alpha_i + \beta_i + \gamma_i} = \prod_{i=1}^n (1 - t_i^2)^{\alpha_i + \beta_i + \gamma_i} = (1 - t^2)^k y^{\alpha + \beta + \gamma} \]

by (3.23).

Now, combining all the observations in the preceding paragraph, we have

\[ M \lambda^m_k(t) = c \lambda^m_k(t) \{ 1 + o(1) \} (1 - t^2)^k F(y) \]
and thus
\[ |M\lambda^m(t)| \gtrsim \lambda^m(t)(1-t^2)^k|cF(y)| = \frac{|cF(y)|}{(1-t^2)^{mn-k}} \]
as \( t \to 1 \). Since \( cF(y) \neq 0 \), we finally conclude
\[ (3.24) \quad |M\lambda^m(t)| \gtrsim 1 \]
as \( t \to 1 \).

**Estimate of \( R_j\lambda^m(t) \):** Recall \( R_j = T_{u_1} \cdots T_{u_{j-1}} T_{e_j} T_{m_{j+1}} \cdots T_{m_4} \) for each \( j \). Let \( j \leq 3 \) for a moment. Using the same argument as in the course of the estimate of \( M\lambda^m(t) \), we have
\[ |T_{m_{j+1}} \cdots T_{m_4}\lambda^m(z)| \lesssim \sum^{(j)} \prod_{i=1}^n S_i^{\alpha_i+\cdots+\gamma_i} \mu_i^m(z_i) \]
where \( \sum^{(j)} \) denote the sum over all \( \alpha \in \Lambda(k_{j+1}), \ldots, \gamma \in \Lambda(k_4). \) Thus we have by Lemma 3.4
\[ |T_{m_{j+1}} \cdots T_{m_4}\lambda^m(z)| \lesssim \sum^{(j)} \prod_{i=1}^n |\mu_i(z_i)|^{m-p_i} \]
where \( p_i = p_i(j; \alpha, \ldots, \gamma) = \alpha_i + \cdots + \gamma_i. \) Hence by (3.20) we have
\[ |T_{e_j} T_{m_{j+1}} \cdots T_{m_4}\lambda^m(z)| \lesssim \sum^{(j)} \Theta_{j, \alpha, \ldots, \gamma}(z) \]
where
\[ \Theta_{j, \alpha, \ldots, \gamma}(z) = \begin{cases} \sum_{\ell=1}^n \int_{D^n} |1 - w_i|^{\gamma} \prod_{i=1}^n \frac{|\mu_i(w_i)|^{m-p_i}}{|1 - z_i w_i|^2} \, dV(w) & \text{if } u_j(e) \neq 0 \\ \sum_{\beta \in \Lambda(k_j)} \int_{D^n} \prod_{i=1}^n \frac{|1 - w_i|^{\beta} |\mu_i(w_i)|^{m-p_i}}{|1 - z_i w_i|^2} \, dV(w) & \text{if } u_j(e) = 0. \end{cases} \]
Recall \( 1 - t_j \approx 1 - t \) as \( t \to 1 \). So, if \( u_j(e) \neq 0 \), then we have by Lemma 3.2
\[ \Theta_{j, \alpha, \ldots, \gamma}(z) \lesssim \sum_{\ell=1}^n \prod_{i \neq \ell} \frac{\Phi_{0,1}(z_i, t_i)}{(1 - t_i)^{m-p_i-2}} \frac{\Phi_{0,1}(z_\ell, t_\ell)}{(1 - t_\ell)^{m-p_\ell-2-\epsilon}} \]
\[ \approx \frac{(1 - t)^\epsilon}{(1 - t)^{mn-2n-(p_1+\cdots+p_n)}} \prod_{i=1}^n \Phi_{0,1}(z_i, t_i) \]
\[ = \frac{(1 - t)^\epsilon}{(1 - t)^{mn-2n-(k_{j+1}+\cdots+k_4)-(k_{j+1})}} \prod_{i=1}^n \Phi_{0,1}(z_i, t_i) \]
as \( t \to 1 \). Similarly, if \( u_j(e) = 0 \), then we have
\[ \Theta_{j, \alpha, \ldots, \gamma}(z) \lesssim \frac{1}{(1 - t)^{mn-2n-(k_{j+1}+\cdots+k_4)-(k_{j+1})}} \prod_{i=1}^n \Phi_{0,1}(z_i, t_i) \]
as $t \to 1$, because $|\beta| = k_j + 1$ for $\beta \in \Lambda^*(k_j)$. Note that these estimates are uniform in $\alpha, \ldots, \gamma$. Thus, since $\epsilon \leq 1$, it follows that

$$|T_{e_j}T_{e_j}T_{e_j} \cdots T_{e_j} \Lambda_4^m(z)| \lesssim \frac{(1-t)^c}{(1-t)^{mn-2n-(k_j+\cdots+k_4)}} \prod_{i=1}^n \Phi_{0,1}(z_i, t_i)$$

as $t \to 1$ and this estimate remains valid even for $j = 4$.

Now, let $j \geq 2$ for a moment. Continuing estimates by using (3.25), we deduce from (3.18) and Lemma 3.2 that

$$|T_{u_{j-1}} T_{e_j} T_{e_j} \cdots T_{e_j} \Lambda_4^m(z)| \lesssim \frac{(1-t)^c}{(1-t)^{mn-2n-(k_j+\cdots+k_4)}} \sum_{\gamma \subseteq \Lambda(k_{j-1})} \left( \prod_{i=1}^n \Phi_{1,2}(z_i, t_i) \right)$$

because $\gamma_i = 0, 1$ for each $i$. Note that, for $\alpha \in \Lambda(k_1), \ldots, \gamma \in \Lambda(k_{j-1})$, we have

$$0 \leq \alpha_i + \cdots + \gamma_i \leq 2$$

for each $i$, because $\alpha = 0$ (recall $k_1 = 0$); it is this step which requires the restriction on the number of factors in the product. Thus, by repeating the same argument using Lemma 3.2, we obtain

$$|T_{u_{j-1}} T_{e_j} T_{e_j} \cdots T_{e_j} \Lambda_4^m(z)| \lesssim \frac{(1-t)^c}{(1-t)^{mn-2n-(k_j+\cdots+k_4)}} \sum_{(\gamma)} \left( \prod_{i=1}^n \Phi_{q_i,j}(z_i, t_i) \right)$$

where $q_i = q_i(j; \alpha, \ldots, \gamma) = \alpha_i + \cdots + \gamma_i$. Here, the sum $\sum_{(\gamma)}$ is taken over all $\alpha \in \Lambda(k_1), \ldots, \gamma \in \Lambda(k_{j-1})$. Since $1 - t_j \approx 1 - t$ as $t \to 1$, each term of the sum above, when evaluated at $t_i$, is estimated as follows:

$$\prod_{i=1}^n \Phi_{q_i,j}(t_i, t_i) \approx \prod_{i=1}^n \frac{1 + |\log(1-t_j)|^j}{(1-t_j)^{2q_i}}$$

$$\lesssim \prod_{i=1}^n \frac{1 + |\log(1-t)|^j}{(1-t)^{2q_i}}$$

$$\lesssim \frac{1 + |\log(1-t)|^j}{(1-t)^{2q_i}}$$

and this estimate is uniform in $\alpha, \ldots, \gamma$. Combining these estimates, we obtain

$$|T_{u_{j-1}} T_{e_j} T_{e_j} \cdots T_{e_j} \Lambda_4^m(t)| \lesssim \frac{(1-t)^c(1 + |\log(1-t)|^j)}{(1-t)^{mn-2n-(k_j+\cdots+k_4)}}, \quad t \to 1$$

for $j \geq 2$ and the same estimate holds for $j = 1$ by a similar argument using (3.25). We thus conclude

$$|R \Lambda_4^m(t)| \lesssim \frac{o(1)}{(1-t)^{mn-2n}}, \quad t \to 1$$

as $t \to 1$. 
Finish of Proof: We easily deduce from (3.24) and (3.27) that
\[ 1 = \frac{|R\lambda_m(t)|}{|M\lambda_m(t)|} \lesssim o(1), \quad t \to 1, \]
which is a contradiction. The proof is complete. \qed

Finally, we prove Theorem 1.3. The proof is exactly the same as that of Theorem 1.2 except at one spot. We only indicate such a difference.

Proof of Theorem 1.3. By a similar argument using Proposition 2.1, we can also have a boundary open set \( W \) where \( u_1 u_2 u_3 \) vanishes and (3.16) holds. In the proof of Theorem 1.2 we were able to assume that \( u_1 \) never vanishes on \( W \) (and thus \( k_1 = 0 \)) under the global Lipschitz hypothesis because the location of the boundary open set \( W \) was of no significance. However, we can not do the same under the present local Lipschitz hypothesis. That is, \( k_1 \geq 1 \) may well happen and the inequality (3.26) is no longer guaranteed. Namely,
\[ \alpha_i + \beta_i + \gamma_i = 3 \]
might hold for some \( i \) and for some \( \alpha \in \Lambda(k_1), \beta \in \Lambda(k_2), \gamma \in \Lambda(k_3) \) in case \( k_1, k_2, k_3 \geq 1 \). What matters here is the number of terms in the sum above which comes from the number of factors. So, if we are given only three factors, all the arguments in the proof of Theorem 1.2 work even in case \( k_1 \geq 1 \). This completes the proof. \qed

REFERENCES