The Essential Spectra of Toeplitz Operators
with Symbols in $H^\infty + C$

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Abstract. The essential spectra of Toeplitz operators acting on the $L^p$-Bergman spaces with continuous symbols have been computed by Zeng. We generalize Zeng’s result to symbols in $H^\infty + C$.

1. Introduction

Let $B$ be the unit ball of the $n$-dimensional complex space $\mathbb{C}^n$. Let $L^p$ denote the Lebesgue space on $B$ endowed with volume measure. The Bergman space $A^p$ ($1 < p < \infty$ is fixed throughout the paper) is then the closed subspace of $L^p$ consisting of holomorphic functions. Let $\mathcal{B}$ be the algebra of all bounded linear operators on $A^p$ and $\mathcal{K}$ be the two-sided compact ideal of $\mathcal{B}$. For an operator $T \in \mathcal{B}$ and a complex number $\lambda$, we say that $\lambda \in \sigma_e(T)$, the essential spectrum of $T$, if $(T - \lambda) + \mathcal{K}$ is not invertible in the Calkin algebra $\mathcal{B}/\mathcal{K}$. In other words, $\lambda \in \sigma_e(T)$ if and only if $T - \lambda$ is not Fredholm.

Let $P$ denote the orthogonal projection from $L^2$ onto $A^2$. As is well-known, the projection $P$ extends to a bounded projection taking $L^p$ onto $A^p$ which can be explicitly described in terms of integration against the Bergman kernel (see, for example, [3, Theorem 7.1.4]). For $u \in L^\infty$, the Toeplitz operator $T_u$ acting on $A^p$ with symbol $u$ is defined by

$$T_u f = P(uf)$$

for $f \in A^p$. It is clear that $T_u \in \mathcal{B}$.

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In [5] Zeng has considered a symbol $u$ continuous up to the boundary and computed the essential spectrum of $T_u$ (Zeng actually worked on the unit disk, but the same argument applies to the ball and, in addition, the argument can now be much simplified by a result of Zhu on the compact Hankel operators mentioned in the next section): $\sigma_e(T_u) = u(S)$ where $S$ denotes the unit sphere, the boundary of $B$. The purpose of the present paper is to generalize this result to symbols in $H^\infty + C$. Here, $H^\infty$ denotes the class of all bounded holomorphic functions on $B$ and $C$ denotes the class of all continuous functions on $\bar{B}$. We prove

**Theorem 1.** Let $u \in H^\infty + C$. Then $\sigma_e(T_u)$ is connected and $\sigma_e(T_u) = \tilde{u}(\beta B \setminus B)$.

The notation $\beta B$ denotes the Stone-Cech compactification of $B$ and $\tilde{u}$ denotes the unique continuous extension of $u$ on $\beta B$.

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2. Proof

We will push the argument of Zeng [5] a little bit further to prove Theorem 1. The following are the ball versions of [5, Theorem 2.4] and [5, Theorem 4.3].

**Lemma 2.** For $u \in C$, we have $T_u \in K$ if and only if $u|_S = 0$. □

**Lemma 3.** For $u \in C$ and $v \in L^\infty$, we have $T_v T_u - T_{uv} \in K$. □

**Remark.** For $u \in L^\infty$, the Hankel operator $H_u$ acting on $A^p$ with symbol $u$ is defined by $H_u f = (I - P)(uf)$ for $f \in A^p$. It is also clear that $H_u \in \mathcal{B}$ and $H_u = 0$ for $u \in H^\infty$. Moreover, it is now known (see [6, Corollary 24]) that for $u \in H^\infty$, we have $H_u \in \mathcal{K}$ if and only if $u$ is in the little Bloch space (for informations on this space see [4]). In particular, we have $H_{\bar{u}} \in \mathcal{K}$ for $u \in H^\infty \cap C$. Since holomorphic and antiholomorphic monomials span
a uniformly dense subset of $C$, we have $H_u \in K$ for $u \in C$. This leads to a simple proof of [5, Theorem 4.3].

**Lemma 4.** For $u \in H^\infty + C$ and $v \in L^\infty$, we have $T_v T_u - T_{uv} \in K$. Moreover, if $v \in H^\infty + C$ as well, then $T_u T_v - T_v T_u \in K$.

**Proof.** Let us write $u = f + \psi$ for some $f \in H^\infty$ and $\psi \in C$. Note that $T_u = T_f + T_\psi$ and $T_v T_f = T_v f$ because $f \in H^\infty$. A simple computation gives $T_v T_u - T_{uv} = T_v T_\psi - T_{v \psi}$, which is compact by Lemma 3, as desired. Moreover, if we have $v \in H^\infty + C$ as well, then $T_u T_v - T_v T_u = (T_u T_v - T_{uv}) - (T_v T_u - T_{vu})$ is also compact by the previous case. This completes the proof. □

To each $u \in H^\infty + C$, there corresponds a boundary function $u^*$ defined by

$$u^*(\zeta) = \lim_{r \to 1} u(r \zeta)$$

at almost all points $\zeta \in S$ with respect to surface area measure on $S$. As is well-known, the map $u \to u^*$ is an isometric isomorphism of $H^\infty$ onto the closed subalgebra of $L^\infty(S)$. Let us write $H^\infty(S)$ for this subalgebra. Then the space $H^\infty(S) + C(S)$ is a closed subalgebra of $L^\infty(S)$. See [3, Theorem 6.5.5]. Clearly, the map $u \to u^*$ is a norm-decreasing algebra homomorphism of $H^\infty + C$ onto $H^\infty(S) + C(S)$. Thus we have a natural Banach algebra isomorphism

(1) \[ [H^\infty + C]/Z \cong H^\infty(S) + C(S) \]

where $Z$ denotes the set of all functions $u \in H^\infty + C$ such that $u^* = 0$.

Let $T$ be the smallest closed algebra generated by all Toeplitz operators $T_u$ with $u \in H^\infty + C$. It is clear that the set $\{T_u + K : u \in H^\infty + C\}$ is dense in $T/K$. Using the same methods as in the Hilbert space case $p = 2$ ([2, Lemma 4.1]), one can estimate the quotient norm $|||T_u + K|||$ of $T_u + K \in T/K$. More explicitly, we have

(2) \[ \limsup_{|z| \to 1} |u(z)| \leq |||T_u + K||| \leq c \limsup_{|z| \to 1} |u(z)| \]
for $u \in H^{\infty} + C$ and for some constant $c = c(n, p)$ depending only on $n$ and $p$ (in fact $c$ is the operator norm of the projection $P : L^p \rightarrow A^p$). At the same time, McDonald [2, Lemma 4.2] computed the essential sup-norm of $u^*$ on $S$:

\begin{equation}
||u^*||_{\infty} = \limsup_{|z| \to 1} |u(z)|.
\end{equation}

for $u \in H^{\infty} + C$. Combining the above with (2), we see that

\begin{equation}
||u^*||_{\infty} \leq ||T_u + K|| \leq c||u^*||_{\infty}
\end{equation}

for $u \in H^{\infty} + C$. We remark in passing that the isomorphism in (1) is actually isometric by (3).

Let us now consider the map $\Lambda : H^{\infty}(S) + C(S) \to T/K$ defined by $\Lambda(u^*) = T_u + K$. To see that $\Lambda$ is well-defined, suppose $u^* = v^*$ for $u, v \in H^{\infty} + C$. Then, by (3), $u - v$ has a continuous extension on $\bar{B}$ with $u - v = 0$ on $S$. It follows that $T_{u - v} = T_u - T_v \in K$ by Lemma 2, or equivalently, $T_u + K = T_v + K$. By (4), the map $\Lambda$ is bounded above and below. Since $\Lambda$ is bounded below, its range is closed. Note that the range of $\Lambda$ is dense by definition of $T$. Therefore the map $\Lambda$ is onto. Moreover, the map $\Lambda$ is an algebra homomorphism by Lemma 4. In summary we obtain from (1)

**Lemma 5.** The map $u + Z \to T_u + K$ defines an algebra isomorphism $[H^{\infty} + C]/Z \cong T/K$. $\Box$

Note that $Z$ is precisely the space of functions continuous on $\bar{B}$ and vanishing on $S$. This follows from (3).

**Lemma 6.** Let $u \in H^{\infty} + C$. Then the following statements are equivalent:

(a) $u + Z$ is invertible in $[H^{\infty} + C]/Z$.

(b) $T_u$ is Fredholm.

(c) $u$ is bounded away from 0 near $S$. 

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Proof. (a) $\Rightarrow$ (b); By assumption and Lemma 5, $T_u + K$ is invertible in $T/K$, hence in $B/K$, as desired.

(b) $\Rightarrow$ (c); This is proved in [2, Theorem 2.5] for $p = 2$. One can show by a similar argument that Lemmas 2.1, 2.2 and 2.3 in [2] are still available for general $p$. Thus this implication holds.

(c) $\Rightarrow$ (a); By assumption and Corollary 4.5 of [2], one sees that $u^*$ is invertible in $H^\infty(S) + C(S)$. This implication now follows from (1). This completes the proof. $\square$

We finally prove our main result.

Proof of Theorem 1. One can see from Lemmas 5 and 6 that $\lambda \in \sigma_e(T_u)$ if and only if $\lambda \in u(B \setminus rB)$ for every $0 < r < 1$. Hence

$$\sigma_e(T_u) = \bigcap_{0 < r < 1} u(B \setminus rB).$$

This shows that $\sigma_e(T_u)$ is the intersection of a nested sequence of compact connected sets and hence connected. Note that $\overline{u(B \setminus rB)} = \hat{u}(\beta B \setminus rB)$ by compactness of $\beta B$ and continuity of $\hat{u}$ on $\beta B$. It follows that

$$\sigma_e(T_u) = \bigcap_{0 < r < 1} \hat{u}(\beta B \setminus rB).$$

Now we show the right side of the above is equal to $\hat{u}(\beta B \setminus B)$. The inclusion $\hat{u}(\beta B \setminus B) \subset \bigcap_{0 < r < 1} \hat{u}(\beta B \setminus rB)$ is clear. Conversely, let $\lambda \in \bigcap_{0 < r < 1} \hat{u}(\beta B \setminus rB)$. Then $\lambda = \hat{u}(z_r)$ for some $z_r \in \beta B \setminus rB$ ($0 < r < 1$). Since $\beta B$ is compact, one can choose a subsequence $r_k$ for which $z_{r_k}$ converges to $z_0$ for some $z_0 \in \beta B$. Moreover, since $B$ is open in $\beta B$, we get $z_0 \in \beta B \setminus B$. Note that $\hat{u}(z_{r_k}) \to \hat{u}(z_0)$ by continuity of $\hat{u}$ on $\beta B$. Since $\lambda = \hat{u}(z_{r_k})$ for each $k$, one obtains $\lambda = \hat{u}(z_0) \in \hat{u}((\beta B \setminus B).$ This shows the inclusion $\bigcap_{0 < r < 1} \hat{u}(\beta B \setminus rB) \subset \hat{u}(\beta B \setminus B)$ and the proof is complete. $\square$

As a simple consequence we recover Zeng’s result [5, Theorem 6.5] on the ball.

**Corollary 7.** Let $u \in C$. Then we have $\sigma_e(T_u) = u(S)$. $\square$
REFERENCES


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