Commuting Toeplitz Operators on the Harmonic Bergman space

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Abstract. We investigate the characterizing problem of harmonic symbols of commuting Toeplitz operators acting on the harmonic Bergman space of the unit disk. We first derive a necessary condition for a certain class of harmonic symbols. As a consequence, we show that Toeplitz operators with holomorphic symbols can commute only in the obvious cases. We prove the same in case at least one of symbols is a (harmonic) polynomial. Also we give a characterization of harmonic symbols of normal Toeplitz operators. The case of general harmonic symbols remains open.

1. Introduction

For $p \geq 1$, we let $L^p = L^p(D, A)$ denote the usual Lebesgue space of the open unit disk $D$ in the complex plane. Here, the letter $A$ denotes the normalized area measure on $D$. The harmonic Bergman space $b^2$ is the subspace of the Lebesgue space $L^2$ consisting of all complex-valued $L^2$-harmonic functions on $D$. One can check the relation $b^2 = L^2_a + \overline{L^2_a}$ where $L^2_a$ denotes the holomorphic Bergman space consisting of all $L^2$-holomorphic functions on $D$. As is well known, the harmonic Bergman space $b^2$ is a closed subspace of $L^2$ and hence is a Hilbert space. We will write $Q$ for the Hilbert space orthogonal projection from $L^2$ onto $b^2$. Each point evaluation is easily verified to be a bounded linear functional on $b^2$. Hence, for each $z \in D$, there exists a unique function $R_z$—called the harmonic Bergman kernel—in $b^2$ which has the following reproducing property:

\begin{equation}
(1) \quad u(z) = < u, R_z >
\end{equation}

for every $u \in b^2$. Here and elsewhere, the notation $< , >$ denotes the usual inner product in $L^2$. Since $b^2 = L^2_a + \overline{L^2_a}$, there is a simple relation between the harmonic Bergman kernel $R_z$ and the well-known (holomorphic) Bergman kernel $K_z$: $R_z = K_z + \overline{K_z} - 1$. Thus, the explicit formula of $R_z$ is given by

\begin{equation}
(2) \quad R_z(w) = \frac{1}{(1 - wz)^2} + \frac{1}{(1 - \overline{w}z)^2} - 1 \quad (w \in D).
\end{equation}
The formulas (1) and (2) lead us to the following integral representation of the projection $Q$:

$$Q\varphi(z) = \int_D \left( \frac{1}{(1-z\bar{w})^2} + \frac{1}{(1-\bar{z}w)^2} - 1 \right) \varphi(w) \, dA(w) \quad (z \in D)$$  \hspace{1cm} (3)

for functions $\varphi \in L^2$. See [ABR, Chapter 8] for more information and related facts.

For $u \in L^2$, the Toeplitz operator $T_u$ with symbol $u$ is defined by

$$T_u f = Q(uf)$$

for functions $f \in b^2$. The operator $T_u$ is densely defined and not bounded in general.

Here, we are concerned with the characterizing problem of harmonic symbols of commuting Toeplitz operators acting on $b^2$. The corresponding problem, as well as its essential version, for the holomorphic Bergman space has been studied by several authors ([AČ], [CL1], [CL2], [L1], [L2], [Z]) and has been completely resolved in case of holomorphic or harmonic symbols. The present harmonic case was first studied by Ohno who observed ([O, Theorem 2.2]) that, for a holomorphic symbol $f$ on $D$, $T_f$ commutes with $T_z$ if and only if $f$ is a polynomial of degree at most 1. In this paper we obtain complete descriptions for certain types of harmonic symbols of commuting Toeplitz operators. All the results we obtained show that Toeplitz operators can commute only in the obvious cases. However, we do not know whether the same is true for general harmonic symbols.

In Section 2 we characterize holomorphic symbols of commuting Toeplitz operators. Our result (Theorem 5) is

- For $f, g \in L^2$, $T_f$ and $T_g$ commute on $b^2$ if and only if a nontrivial linear combination of $f$ and $g$ is constant on $D$.

For this result, we consider a more general symbols and derive a necessary condition for those symbols to induce commuting Toeplitz operators (Theorem 4):

- Let $u, v \in b^2$. Suppose $T_u$ and $T_v$ commute on $b^2$. If $u, v$ are both not antiholomorphic, then there exists a constant $\alpha$ such that $\partial v = \alpha(\partial u)$ where $\partial = \partial/\partial z$.

This result also plays the key role in proving our results in the next section.

In Section 3 we consider two special types of symbols and prove characterizations for those symbols. We first consider a pair of symbols related to each other by complex conjugation and give a characterization for those symbols. This might be of some independent interest, because they are just symbols of normal Toeplitz operators. Recall that a bounded linear operator on a Hilbert space is called normal if it commutes with its adjoint operator. Our characterization shows that only obvious ones are normal. Then we consider a pair of symbols in case one of them is a (harmonic) polynomial. Our results (Theorem 8, Theorem 10) are

- For $u \in b^2$, $T_u$ is normal on $b^2$ if and only if $u(D)$ is contained in a straight line.
- Let $u$ be a (harmonic) polynomial. For $v \in b^2$, $T_u$ and $T_v$ commute on $b^2$ if and only if a nontrivial linear combination of $u$ and $v$ is constant on $D$. 

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2. Holomorphic Symbols

In this section we give a characterization of holomorphic symbols of commuting Toeplitz operators (Theorem 5). For that purpose, we consider a bit more general symbols having nonconstant holomorphic parts and derive a necessary condition (Theorem 4). This necessary condition is also the key to the proofs of results in the next section.

Before proceeding, let us recall the well-known Bergman projection \( P \). For each \( z \in D \), the explicit formula for the Bergman kernel \( K_z \) is given by

\[
K_z(w) = \frac{1}{(1 - wz)^2} \quad (w \in D)
\]

and the Bergman projection \( P \) is the integral operator

\[
Pu(z) = \langle u, K_z \rangle = \int_D \frac{u(w)}{(1 - z\bar{w})^2} dA(w)
\]

taking \( L^1 \)-functions \( u \) into the space of all holomorphic functions. As is well known (see, for example, [Zh, Chapter 4]), the Bergman projection \( P \), when restricted to \( L^2 \), is the Hilbert space orthogonal projection from \( L^2 \) onto \( L^2_a \). Moreover, \( P \) has the following reproducing property:

\[
(4) \quad Pf = f \quad \text{and} \quad P\bar{f} = \bar{f}(0)
\]

for all holomorphic \( L^1 \)-functions \( f \).

We start with some simple properties of the Bergman projection which will be useful in the proofs.

**Lemma 1.** Let \( f \in L^2_a \). Then the following statements hold for all \( z \in D \).

\[
(a) \quad \frac{d}{dz} \left\{ z^2P[(1 - |w|^2)f] \right\} = zf(z).
\]

\[
(b) \quad \frac{d}{dz} \left\{ z^2P[(1 - |w|^2)f] \right\} = z\bar{f}(0).
\]

\[
(c) \quad P(|w|^2f)(z) = f(z) - \frac{1}{z^2} \int_0^z \zeta f(\zeta) d\zeta.
\]

\[
(d) \quad P(|w|^2\bar{f}) = \overline{P(|w|^2f)(0)} = \frac{1}{2}f(0).
\]

**Proof.** For \( \psi = f \) or \( \psi = \bar{f} \), let \( g = P[(1 - |w|^2)\psi] \). That is,

\[
g(z) = \int_D \frac{\psi(w)}{(1 - z\bar{w})^2} (1 - |w|^2) dA(w) \quad (z \in D).
\]

Then, by differentiating under the integral sign, we have

\[
g'(z) = 2\int_D \frac{\psi(w)\bar{w}}{(1 - z\bar{w})^3} (1 - |w|^2) dA(w)
\]
so that
\[
\frac{d}{dz} \{z^2 g(z)\} = 2zg(z) + z^2 g'(z) = 2z \int_D \frac{\psi(w)(1 - |w|^2)}{(1 - zw)^3} dA(w)
\]
for all \( z \in D \). Now, (a) and (b) follow from Theorem 1 of [C].

Note from (4) that
\[
z^2 P[(1 - |w|^2)f](z) = z^2 f(z) - z^2 P(|w|^2 f)(z)
\]
for all \( z \in D \). Thus integrating both sides of (a), we have
\[
z^2 f(z) - z^2 P(|w|^2 f)(z) = \int_0^z \zeta f(\zeta) d\zeta \quad (z \in D)
\]
which implies (c). Also, we have by (4)
\[
z^2 P[(1 - |w|^2)\bar{f}](z) = z^2 P(\bar{f}) - z^2 P(|w|^2 \bar{f}) = z^2 \bar{f}(0) - z^2 P(|w|^2 \bar{f})
\]
and thus, integrating both sides of (b) shows
\[
P(|w|^2 \bar{f}) = \frac{1}{2} \bar{f}(0).
\]
Since
\[
P(|w|^2 f)(0) = \int_D f(w)|w|^2 dA(w) = f(0) \int_D |w|^2 dA(w) = \frac{1}{2} f(0),
\]
we also have
\[
\frac{P(|w|^2 f)(0)}{1} = \frac{1}{2} \bar{f}(0).
\]
This proves (d). The proof is complete. \( \Box \)

The following two lemmas will much simplify our arguments in the proofs later.

**Lemma 2.** Let \( f \in L^2_a \) and suppose \( f(0) = 0 \). Then the following statements hold for all \( z \in D \).

(a) \( P(\bar{w}f)(z) = \frac{1}{z} f(z) - \frac{1}{z^2} \int_0^z f(\zeta) d\zeta \).

(b) \( P(w\bar{f})(z) = P(\bar{w}f)(0) = \frac{1}{2} f'(0) \).

**Proof.** Since \( f(0) = 0 \), there is a holomorphic function \( g \) on \( D \) such that \( f(z) = zg(z) \). One can easily check that \( g \) is in \( L^2_a \). Now, by Lemma 1, we have
\[
P(\bar{w}f)(z) = P(|w|^2 g)(z) = g(z) - \frac{1}{z^2} \int_0^z \zeta g(\zeta) d\zeta \quad (z \in D)
\]
which gives (a). Similarly, we have
\[
P(w\bar{f}) = P(|w|^2 \bar{g}) = \frac{1}{2} \bar{g}(0)
\]
and thus (b) holds. The proof is complete. \( \Box \)
Lemma 3. Let $f, g \in L^2_a$. Then the following statements hold.

(a) $P\left(\overline{fP(\bar{w}g)}\right)(z) = P(f\bar{w}g)(z)$ for each $z \in D$.

(b) $P\left(f\overline{P(\bar{w}g)}\right)(0) = P(f\bar{w}g)(0)$.

Proof. Since the Bergman projection $P$ is the orthogonal projection from $L^2$ onto $L^2_a$, we see from (4)

$$P\left(\overline{fP(\bar{w}g)}\right)(z) = \langle \overline{fP(\bar{w}g)}, K_z \rangle = \langle P(\bar{w}g), fK_z \rangle = P(f\bar{w}g)(z)$$

for each $z \in D$, so we have (a). By a similar argument, one can also see that

$$P\left(f\overline{P(\bar{w}g)}\right)(0) = \int_D f\overline{P(\bar{w}g)} \, dA = \langle f, P(\bar{w}g) \rangle = \langle f, \bar{w}g \rangle = P(f\bar{w}g)(0),$$

so (b) holds. The proof is complete. □

Since $P\varphi(0) = \int_D \varphi \, dA$ for functions $\varphi \in L^2$, we see from (3) that the projection $Q$ can be rewritten as

$$Q\varphi = P(\varphi) + \overline{P(\bar{\varphi})} - P(\varphi)(0)$$

for functions $\varphi \in L^2$. Now, we prove the following.

Theorem 4. Let $u, v \in b^2$. Suppose $T_u$ and $T_v$ commute on $b^2$. If $\partial u$ and $\partial v$ are both not identically 0, then there exists a constant $\alpha$ such that $\partial v = \alpha \partial u$.

For holomorphic functions $f$ and $g$, we will use the fact that $f + \bar{g} \in b^2$ implies $f, g \in L^2_a$. A proof is given here for reader’s convenience. Assume $g(0) = 0$ for simplicity. Put $u = f + \bar{g}$ and let $u_r(z) = u(rz)$ for $z \in D$ and $0 < r < 1$. By (4), we have $P(u_r) = f_r$. Since $P$ is bounded on $L^2$, taking the limit $r \to 1$, we have $P(u) = f \in L^2_a$, and thus $g = \bar{u} - f \in L^2_a$. In fact, a little bit more is true: If $f, g$ are holomorphic functions such that $f + \bar{g} \in L^p$ for $p \geq 1$, then $f, g \in L^p$. See the proof of Theorem 7.1.5 of [R].

Proof. By the remark above, there are functions $f, g, h$ and $k$ in $L^2_a$ such that $u = f + \bar{g}$ and $v = h + k$. Without loss of generality, we may assume $f(0) = g(0) = h(0) = k(0) = 0$. By assumption, $f$ and $h$ are nonconstant. We need to show $h = \alpha f$ for some constant $\alpha$.

By (5) and Lemma 2, we have

$$T_h(\bar{w}) = Q(\bar{wh}) = P(\bar{w}h) + \overline{P(wh)} - P(\bar{wh})(0) = P(\bar{w}h) + P(\bar{wh})(0) - P(\bar{wh})(0) = P(\bar{w}h)$$
and hence

$$T_fT_h(\bar{w}) = fP(\bar{w}h).$$

Also, since $Q(\bar{w}k) = \bar{w}k$, we have

$$T_fT_k(\bar{w}) = Q[fQ(\bar{w}k)]$$

$$= Q(f\bar{w}k)$$

$$= P(f\bar{w}k) + P(fwk) - P(f\bar{w}k)(0).$$

Next, by (6) and Lemma 3, we have

$$T_gT_h(\bar{w}) = Q[\bar{g}P(\bar{w}h)]$$

$$= P[\bar{g}P(\bar{w}h)] + P[\bar{g}P(\bar{w}h)] - P[\bar{g}P(\bar{w}h)](0)$$

$$= P(\bar{g}\bar{w}h) + P[\bar{g}P(\bar{w}h)] - P(\bar{g}\bar{w}h)(0)$$

and, finally note that

$$T_gT_k(\bar{w}) = \bar{g}\bar{w}k.$$

It follows that

$$T_f + \bar{g}T_h + \bar{w}k(\bar{w}) = T_h + \bar{w}kT_f + \bar{g}(\bar{w})$$

$$= fP(wh) + P(fwk) + P(fwk) - P(f\bar{w}k)(0)$$

$$+ P(\bar{g}\bar{w}h) + P[\bar{g}P(\bar{w}h)] - P(\bar{g}\bar{w}h)(0) + \bar{g}\bar{w}k.$$

By exactly the same way,

$$T_{h+k} = hP(\bar{w}f) + P(h\bar{w}g) + P(hw) - P(h\bar{w}g)(0)$$

$$+ P(\bar{w}f) + P[kP(\bar{w}f)] - P(\bar{w}f)(0) + \bar{w}g.$$
for all $z \in D$. Consequently, letting

$$H(z) = \int_{0}^{z} h(\zeta) \, d\zeta, \quad F(z) = \int_{0}^{z} f(\zeta) \, d\zeta,$$

we have $fH = hF$ on $D$. Hence $F'H = H'F$ because $H' = h$ and $F' = f$. Now, since $f$ and $h$ are nonconstant, we see $H = \alpha F$ for some constant $\alpha$. Consequently, we have $h = \alpha f$ for some constant $\alpha$, as desired. This completes the proof. □

As an immediate consequence of Theorem 4, we obtain a complete description of holomorphic symbols of commuting Toeplitz operators.

**Theorem 5.** Let $f, g \in L^2_a$ be nonconstant functions. Then $T_f T_g = T_g T_f$ on $b^2$ if and only if $g = \alpha f + \beta$ for some constants $\alpha, \beta$.

**Proof.** This is immediate from Theorem 4. □

### 3. Two Special Cases

In this section we give characterizations for a pair of symbols of special type to induce commuting Toeplitz operators. We consider two types. One is a pair of symbols related to each other by complex conjugation. This type of symbols might be of some independent interest, because it is just that of normal Toeplitz operators. The other is a pair of symbols for which at least one is a polynomial. Our results (Theorem 8, Theorem 10) show that such symbols must be related in an obvious way, as expected.

We begin with an integral identity which is taken from [C].

**Lemma 6.** For $f, g \in L^2_a$, we have

$$\int_{D} f(w) \overline{g(w)} \, dA(w) = \int_{D} f(w) [2g(w) + \overline{wg'(w)}] (1 - |w|^2) \, dA(w).$$

**Proof.** See Theorem 12 of [C] where the lemma is stated for slightly different pairs of $f$ and $g$. □

The following lemma is useful for our purposes.

**Lemma 7.** Let $f, g \in L^2_a$ and assume $f(0) = g(0) = 0$. If $T_f$ and $T_g$ commute on $b^2$, then

$$\int_{D} f(w) \overline{G(w)w^k} (1 - |w|^2) \, dA(w) = 0 \quad (k = 0, 1, 2, \ldots)$$

where

$$G(w) = \frac{1}{w} \int_{0}^{w} g(\zeta) \, d\zeta.$$

**Proof.** Since $T_f T_g = T_g T_f$, we have $T_g T_f = T_f T_g$ by taking adjoints. In particular, we have $T_g T_f(\bar{w}) = T_f T_g(\bar{w})$. Now, repeating exactly the same argument as in the proof of Theorem 4 yields

$$P[fP(\bar{g})] = P(fw\bar{g})$$
so that
\[
< f, P(g\bar{w})w^{k+1} > = < P[fP(g\bar{w})], w^{k+1} > = < P(fw\bar{g}), w^{k+1} > = < fw, gw^{k+1} >
\]
for all \( k \geq 0 \). Rearranging the above by using Lemma 2, we have
\[
\int_D f(w)g(w)w^k (1 - |w|^2) dA(w) = \int_D f(w)G(w)w^k dA(w).
\]
On the other hand, by Lemma 6,
\[
\int_D f(w)\overline{G(w)}w^k dA(w) = \int_D f(w)[G'(w)w + (k + 2)G(w)]w^k (1 - |w|^2) dA(w)
\]
and thus, by (8),
\[
\int_D f(w)\overline{G(w)}w^k (1 - |w|^2) dA(w) = 0
\]
for all \( k \geq 0 \). The proof is complete. \( \square \)

We turn to the characterization of harmonic symbols of normal Toeplitz operators. It shows that only harmonic symbols of normal Toeplitz operators are obvious ones.

**Theorem 8.** Let \( u \in b^2 \). Then \( T_u \) is normal on \( b^2 \) if and only if \( u(D) \) is contained in a straight line. In particular, for \( f \in L^2_{a} \), \( T_f \) is normal on \( b^2 \) if and only if \( f \) is constant.

**Proof.** Throughout the proof we assume \( u(0) = 0 \) without loss of generality. Suppose \( u(D) \) is contained in a straight line. Then there exists a holomorphic function \( f \in L^2 \) such that \( u = \alpha(f + \bar{f}) \) for some constant \( \alpha \). Now one can easily check that \( T_u \) is normal.

Conversely, assume \( T_u \) is normal. First consider the case when \( u \) is holomorphic and use the temporary notation \( u = f \). Since the adjoint operator of Toeplitz operator \( T_f \) is \( T_{\bar{f}} \), we have \( T_f T_{\bar{f}} = T_{\bar{f}} T_f \). Thus, setting
\[
F(w) = \frac{1}{w} \int_0^w f(\zeta) \, d\zeta,
\]
we see from Lemma 7 that
\[
0 = \int_D f(w)\overline{F(w)}(1 - |w|^2) \, dA(w)
\]
\[
= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{F^{(n)}(0)}{n!} \int_D |w|^{2n}(1 - |w|^2) \, dA(w).
\]
Note that
\[
F^{(n)}(0) = \frac{f^{(n)}(0)}{n + 1} \quad (n = 1, 2, \ldots).
\]
Insert these into (9) and get

\[ \sum_{n=1}^{\infty} \frac{|f^{(n)}(0)|^2}{n!} \cdot \frac{1}{n+1} \int_{D} |w|^{2n}(1-|w|^2) \ dA(w) = 0, \]

which means \( f^{(n)}(0) = 0 \) for all \( n \geq 1 \). Thus \( f \) is identically 0, as desired.

Now, consider general \( u = f + \tilde{g} \) where \( f, g \in L^2_{\alpha} \) are nonconstant functions with \( f(0) = g(0) = 0 \). Then, by Theorem 4, there is a constant \( \alpha \) such that \( g = \alpha f \). Now, normality of \( T_u \) is equivalent to

\[ (1-|\alpha|^2)(T_f T_f - T_f T_f) = 0. \]

Since \( f \) is nonconstant, \( T_f \) is not normal by the previous case and thus the above yields \( |\alpha| = 1 \). Consequently, \( \tilde{u} = \alpha u \) so that \( \sqrt{\alpha}u \) is real-valued. This completes the proof. \( \square \)

Before characterizing harmonic symbols of commuting Toeplitz operators in case one of their symbols is a polynomial, we first prove the following special case.

**Lemma 9.** Let \( f, g \in L^2_{\alpha} \) and suppose one of them is a polynomial. If \( T_f \) and \( T_{\tilde{g}} \) commute on \( b^2 \), then either \( f \) or \( g \) is constant.

**Proof.** We assume that both \( f \) and \( g \) are nonconstant and derive a contradiction. Note that the adjoint operators \( T_{\overline{f}} \) and \( T_{\overline{g}} \) also commute. Thus, we may assume \( f \) is a polynomial of degree \( n \geq 1 \) without loss of generality. We may further assume \( f(0) = g(0) = 0 \). Since \( g \) is nonconstant, there is a function \( h \in L^2_{\alpha}, h(0) \neq 0 \), such that \( g = w^k h \) for some positive integer \( k \).

Consider the case \( k \leq n \). Denote \( G \) for the function associated with \( g \) as in Lemma 7. Then it follows from Lemma 7 that

\[ 0 = \int_{D} f(w)G(w)w^{n-k}(1-|w|^2) \ dA(w) = \frac{f^{(n)}(0)}{n!} \cdot \frac{G^{(k)}(0)}{k!} \int_{D} |w|^{2n}(1-|w|^2) \ dA(w) \]

so that

\[ h(0) = \frac{g^{(k)}(0)}{k!} = (k+1) \cdot \frac{G^{(k)}(0)}{k!} = 0, \]

which is a contradiction.

Consider the case \( k \geq n+1 \). Since \( T_f \) and \( T_{\overline{g}} \) commute, we have \( T_f T_{\overline{g}}(\tilde{w}^{n+1}) = T_{\overline{g}} T_f(\tilde{w}^{n+1}) \). Before calculating these, note

\[ P(\tilde{w}^i w^j) = \begin{cases} 0 & \text{for } i > j \\ \frac{j+1-i}{j+1} w^{j-i} & \text{for } i \leq j \end{cases} \]

Before characterizing harmonic symbols of commuting Toeplitz operators in case one of their symbols is a polynomial, we first prove the following special case.
where \(i, j\) are nonnegative integers. This is easily verified by a straightforward calculation. Since \(f\) is a polynomial of degree \(n\), it follows from (10)

\[
P(f \bar{g} w^{n+1}) = P(f \bar{w}^{n+1}) = 0.
\]

Now, as in the proof of Theorem 4, we have

\[
T_f T_g(\bar{w}^{n+1}) = P(f \bar{g} w^{n+1}) + \overline{P(f g w^{n+1})} - P(f \bar{g} w^{n+1})(0)
= P(f g w^{n+1})
\]

and

\[
T_g T_f(\bar{w}^{n+1}) = T_g Q(f \bar{w}^{n+1})
= T_g[P(f \bar{w}^{n+1}) + \overline{P(f w^{n+1})}] - P(f \bar{w}^{n+1})(0)
= gP(f w^{n+1})
\]

so that

\[
P(\bar{f} g w^{n+1}) = gP(\bar{f} w^{n+1}).
\]

In particular, we have

\[
< P(\bar{f} g w^{n+1}), w^{k+1} > = < gP(\bar{f} w^{n+1}), w^{k+1} >.
\]

On one hand, since \(f w^{k+1}\) is a polynomial of degree \(n + k + 1\), we see

\[
< P(\bar{f} g w^{n+1}), w^{k+1} > =< \bar{f} g w^{n+1}, w^{k+1} >
= < h w^{n+k+1}, f w^{k+1} >
= h(0) \cdot \frac{f(n)(0)}{n!} \int_D |w|^{2(n+k+1)} dA(w)
= h(0) \cdot \frac{f(n)(0)}{n!(n + k + 2)}.
\]

On the other hand, since

\[
P(\bar{g} w^{k+1}) = P(\bar{h} w^{k+1})
= \frac{1}{k + 2} [2h(0)w + h'(0)]
\]
by (10) and

\[ < \bar{f} w^{n+1}, 1 > = < w^{n+1}, f > = 0 \]
\[ < \bar{f} w^{n+1}, w > = < w^{n+1}, f w > = \frac{f(n)(0)}{n!(n+2)}, \]

it follows that

\[ < g P(\bar{f} w^{n+1}), w^{k+1} > = < P(\bar{f} w^{n+1}), \bar{g} w^{k+1} > \]
\[ = < \bar{f} w^{n+1}, P(\bar{g} w^{k+1}) > \]
\[ = \frac{2h(0)}{k+2} \cdot \frac{f(n)(0)}{n!(n+2)}. \]

In summary, we have

\[ \frac{h(0)}{2} \cdot \frac{f(n)(0)}{n+k+2} = \frac{h(0)}{k+2} \cdot \frac{f(n)(0)}{n+2}. \]

One can check that this is also impossible. The proof is complete. \( \square \)

We now give the following characterization showing that Toeplitz operators can commute only in the obvious cases if at least one of their symbols is a polynomial.

**Theorem 10.** Let \( u, v \in b^2 \) be nonconstant functions and suppose one of them is a polynomial. Then \( T_u \) and \( T_v \) commute on \( b^2 \) if and only if \( v = \alpha u + \beta \) for some constants \( \alpha, \beta \).

**Proof.** The sufficiency is trivial. For the proof of necessity, assume \( u(0) = v(0) = 0 \) without loss of generality. We split the proof into cases. Since the adjoint operators \( T_u \) and \( T_v \) also commute, we may assume that \( u \) is a polynomial and need to consider only two cases as follows. The remaining cases are contained in Theorem 5 and Lemma 9.

(Case 1) Consider the case when \( u \) is holomorphic and \( v \) is not antiholomorphic. In this case, by Theorem 4, there is a constant \( \alpha \) such that \( \partial v = \alpha (\partial u) \). Accordingly, we may write \( u = f \) and \( v = \alpha f + \bar{g} \) for some \( f, g \in L^2_a \). Thus, the assumption \( T_u T_v = T_v T_u \) is equivalent to

\[ T_f T_{\bar{g}} - T_{\bar{g}} T_f = 0. \]

That is, \( T_f \) and \( T_{\bar{g}} \) commute. Since \( f \) is nonconstant, it follows from Lemma 9 that \( g \) must be constant (identically 0) and thus \( v = \alpha u \).

(Case 2) Consider the case when both \( u \) and \( v \) are neither holomorphic nor antiholomorphic. In this case, by Theorem 4, there are constants \( \alpha, \beta \) such that \( \partial v = \alpha (\partial u), \partial v = \beta (\partial u) \). Accordingly, we may write \( u = f + \bar{g}, v = \alpha f + \beta \bar{g} \) for some nonconstant \( f, g \in L^2_a \) with \( f(0) = g(0) = 0 \). Thus, the assumption \( T_u T_v = T_v T_u \) is equivalent to

\[ (\alpha - \beta) (T_f T_{\bar{g}} - T_{\bar{g}} T_f) = 0. \]
Since $f$ and $g$ are nonconstant, it follows from Lemma 9 that $\alpha = \beta$ and hence $v = \alpha u$. The proof is complete. □

In view of theorems proved in this paper, one may naturally ask whether the same is true for general harmonic symbols. So, we close the paper with a question. To be more precise, let $u, v \in b^2$ be nonconstant functions. We do not know whether $T_u T_v = T_v T_u$ implies $v = \alpha u + \beta$ for some constants $\alpha, \beta$. As we have seen in the proof of Theorem 10, this problem reduces to the following special case:

**Problem.** Let $f, g \in L^2_a$ and suppose $T_f$ and $T_g$ commute on $b^2$. Does it follow that either $f$ or $g$ is constant?

**References**


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