DERIVATIVES OF HARMONIC BERGMAN AND BLOCH FUNCTIONS ON THE BALL

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ABSTRACT. On the setting of the unit ball of euclidean n-space, we investigate properties of derivatives of functions in the harmonic Bergman space and the harmonic Bloch space. Our results are (1) size estimates of derivatives of the harmonic Bergman kernel, (2) Gleason’s problem, and (3) characterizations in terms of radial, tangential and ordinary derivative norms. In the course of proofs, some reproducing formulas are found and estimated.

1. Introduction

For a fixed positive integer \( n \geq 2 \), let \( B \) be the open unit ball in \( \mathbb{R}^n \). The harmonic Bergman space \( b^p \), \( 1 \leq p < \infty \), is the space of all harmonic functions \( f \) on \( B \) such that

\[
||f||_p = \left( \int_B |f|^p \, dV \right)^{1/p} < \infty
\]

where \( V \) is the volume measure on \( B \). The space \( b^p \) is a closed subspace of \( L^p(B, dV) \) and thus a Banach space.

The harmonic Bloch space \( B \) is the space of harmonic functions \( f \) on \( B \) with the property that the function \((1 - |x|^2)||\nabla f(x)||\) is bounded on \( B \). The space \( B \) is also a Banach space equipped with norm

\[
||f||_B = |f(0)| + \sup_{x \in B} (1 - |x|^2)||\nabla f(x)||.
\]

The harmonic little Bloch space \( B_0 \) is the space of harmonic functions \( f \in B \) with the additional property that \((1 - |x|^2)||\nabla f(x)||\) is vanishing on \( \partial B \).

The main results of this paper are Theorems 1.1, 1.2, 1.3 and 1.4 below. For a given multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with each \( \alpha_j \) a nonnegative integer, we use notations \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) where \( \partial_j \) denotes the differentiation with respect to \( j \)-th variable. We sometimes attach a variable subscript to indicate the specific variable with respect to which differentiation is to be taken.

**Theorem 1.1.** Let \( R(x, y) \) denote the harmonic Bergman kernel for \( B \). Given multi-indices \( \alpha \) and \( \beta \), there exists a positive constant \( C = C(\alpha, \beta) \) such that

\[
|\partial_\alpha \partial_\beta \frac{R(x, y)}{(1 - 2x \cdot y + |x|^2 |y|^2)^{(n+|\alpha|+|\beta|)/2}}| \leq C
\]

for all \( x, y \in B \).

It is the above estimate which enables us to obtain most of the following results.

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Theorem 1.2. For any positive integer $m$ and multi-indices $\alpha$ with $|\alpha| = m$, there exist linear operators $A_\alpha$ on harmonic functions in $B$ with the following property:

1. If $f$ is harmonic on $B$ and all its partial derivatives up to order $m - 1$ vanish at 0, then
   \[ f(x) = \sum_{|\alpha| = m} x^\alpha A_\alpha f(x) \quad (x \in B). \]

2. Each $A_\alpha$ is bounded on $B_p$ for $1 < p < \infty$.
3. Each $A_\alpha$ is bounded on $B$ and $B_0$.

We let $Df(x) = x \cdot \nabla f(x)$ denote the radial derivative of $f$. Also, we let $T^\alpha f$ denote tangential derivatives of $f$ (see Section 5). In terms of derivatives, the harmonic Bergman spaces are characterized in the following way.

Theorem 1.3. Let $1 \leq p < \infty$ and $m$ be a positive integer. Then, for a function $f$ harmonic on $B$, the following conditions are equivalent.

1. $f \in \mathcal{B}_{p}$.
2. $(1 - |x|^2)^{m} D^m f \in \mathcal{L}^p$.
3. $(1 - |x|^2)^{m} T^\alpha f \in \mathcal{L}^p$ for all $\alpha$ with $|\alpha| = m$.
4. $(1 - |x|^2)^{m} \partial^\alpha f \in \mathcal{L}^p$ for all $\alpha$ with $|\alpha| = m$.

As a companion result for the harmonic Bloch space, we have the following. Here, $C_0 = C_0(B)$.

Theorem 1.4. Let $m$ be a positive integer. Then, for a function $f$ harmonic on $B$, the following conditions are equivalent.

1. $f \in \mathcal{B} (B_0 \text{ resp.})$.
2. $(1 - |x|^2)^{m} D^m f \in \mathcal{L}^\infty (C_0 \text{ resp.})$.
3. $(1 - |x|^2)^{m} T^\alpha f \in \mathcal{L}^\infty (C_0 \text{ resp.})$ for all $\alpha$ with $|\alpha| = m$.
4. $(1 - |x|^2)^{m} \partial^\alpha f \in \mathcal{L}^\infty (C_0 \text{ resp.})$ for all $\alpha$ with $|\alpha| = m$.

It seems worth to compare and contrast Theorem 1.3 and Theorem 1.4 with known results for Bergman spaces and Bloch spaces of holomorphic functions. Analogous results (only for ordinary derivative norms) for the holomorphic case are proved in [8]. A significant difference that we wish to point out here is that tangential and radial growth of harmonic Bergman and Bloch functions are of the same order; this is not the case for holomorphic functions where additional smoothness occurs in the complex tangential directions (see Section 6.4 of [4] or [7]).

In Section 2 we first estimate sizes of derivatives of the harmonic Bergman kernel. Then we obtain a couple of estimates which show how integrals of the harmonic Bergman kernel behave near boundary. In Section 3 we prove a couple of reproducing formulas and estimate them as a preliminary step towards results in later sections. In Section 4 we give proofs of solutions to Gleason’s problem. Based on the estimates in the previous section, the proofs are quite simple. Finally, in Section 5, we prove characterizations in terms of derivatives. The radial, tangential and ordinary derivatives are all considered. We introduce corresponding norms and show that all the norms are equivalent. Here, we employ a direct approach to estimate norms, which are quite different from previous ones used in some other settings.

Constants. Throughout the paper we will use the same letter $C$ to denote various constants which may change at each occurrence. For nonnegative quantities $A$ and
By the mean value property of harmonic functions, it is easily seen that point evaluations are continuous on $B^2$. Thus, to each $x \in B$, there corresponds a unique $R(x, \cdot) \in \mathcal{B}^2$ such that
\begin{equation}
\tag{2.1}
f(x) = \int_B f(y)R(x, y)\,dy
\end{equation}
for functions $f \in \mathcal{B}^2$ and thus for $f \in B^1$, because $\mathcal{B}^2$ is dense in $B^1$. Here and elsewhere, we let $dy = dV(y)$. It is well known that the kernel function is real and hence the complex conjugation in the integral of (2.1) can be removed. The explicit formula of the kernel function is also well known (see, for example, [1]):
\[ R(x, y) = \frac{1}{nV(B)\rho^2(x, y)} \left\{ \rho^2 - 4|x|^2|y|^2 \right\}, \]
where
\[ \rho(x, y) = \sqrt{1 - 2x \cdot y + |x|^2|y|^2}. \]
Since $1 - |x||y| \leq \rho(x, y)$, it is clear that
\begin{equation}
\tag{2.2}
|R(x, y)| \lesssim \rho^{-n}(x, y)
\end{equation}
for all $x, y \in B$. The key step to our results is the optimal size estimates of derivatives of the reproducing kernel in terms of $\rho$. Note that a standard argument using Cauchy’s estimates cannot be directly applied. We first prove a lemma.

**Lemma 2.1.** Given multi-indices $\alpha$, $\beta$, and $\epsilon > 0$, there exists a positive constant $C = C(\alpha, \beta, \epsilon)$ such that
\[ |\partial_\alpha^\beta \rho^{-\epsilon}(x, y)| \leq \frac{C}{\rho^{\epsilon+|\alpha|+|\beta|}(x, y)}, \]
whenever $x, y \in B$, $|\alpha - \beta| \geq (1 - |x|)/2$ and $|x - y| \geq (1 - |y|)/2$.

**Proof.** Fix $x, y \in B$ such that $|\alpha - \beta| \geq (1 - |x|)/2$ and $|x - y| \geq (1 - |y|)/2$. We let $\rho = \rho(x, y)$. Note
\begin{equation}
\tag{2.3}
\rho^2 = (1 - |x|^2)(1 - |y|^2) + |x - y|^2.
\end{equation}
Clearly we have
\begin{equation}
\tag{2.4}
|\partial_\alpha^\beta \rho^{-\epsilon}| \lesssim 1 \lesssim \rho^{-2(|\alpha|+|\beta|)}
\end{equation}
for $|\alpha| + |\beta| \geq 2$. Also, note $\nabla_x \rho^2 = -2y + 2x|y|^2$ and $\nabla_y \rho^2 = -2x + 2y|x|^2$. Thus,
\[ |\nabla_x \rho^2|^2 + |\nabla_y \rho^2|^2 \lesssim |x - y| + (1 - |x|) + (1 - |y|) \lesssim \rho \]
so that (2.4) remains true for all $\alpha$ and $\beta$.

First, we prove the lemma for $\epsilon = 2$. We prove by induction on $|\alpha| + |\beta|$. There is nothing to prove for $|\alpha| + |\beta| = 0$. Let $|\alpha| + |\beta| = k$ and assume the lemma is
true for all $\alpha', \beta'$ with $|\alpha'| + |\beta'| \leq k$. Let $\partial_j = \partial_{j,x}$ or $\partial_{j,y}$. Then, by (2.4) and induction hypothesis, we have

$$\left| \partial_x^p \partial_y^q \partial_j \rho^{-2} \right| = \left| \partial_x^p \partial_y^q \left( \rho^{-1} \partial_j \rho^2 \right) \right|$$

$$\lesssim \sum \left| \partial_x^p \partial_y^q \partial_x^{\alpha'} \partial_y^{\beta'} \rho \right| \left| \partial_x^{\alpha''} \partial_y^{\beta''} \left( \rho^{-2} \partial_j \rho^2 \right) \right|$$

$$\lesssim \sum \rho^{1-|\alpha'|-|\beta'|-|\alpha''|-|\beta''|} \rho^{-2-|\alpha'|-|\beta'|-|\alpha''|-|\beta''|}$$

$$\approx \rho^{3-k}$$

where the sum is taken over all $\alpha', \alpha'', \beta', \beta''$ such that $|\alpha'| + |\alpha''| + |\beta'| + |\beta''| = |\alpha|$ and $|\beta'| + |\beta''| = |\beta|$. This completes the proof for $\epsilon = 2$.

For general $\epsilon > 0$, note

$$\left| \partial_x^p \partial_y^q \partial_j \rho^{-2} \right| = \left| \partial_x^p \partial_y^q \left( \rho^{-\epsilon-2} \partial_j \rho^2 \right) \right|$$

$$\lesssim \sum \left| \partial_x^p \partial_y^q \partial_x^{\alpha'} \partial_y^{\beta'} \rho^{-\epsilon-1} \partial_x^{\alpha''} \partial_y^{\beta''} \left( \rho^{-2} \partial_j \rho^2 \right) \right|$$

Thus, using the result for $\epsilon = 2$, one may complete the proof by induction as above.

The following lemma is a consequence of the mean value property of harmonic functions and Cauchy’s estimate. For details, see Corollary 8.2 of [1].

**Lemma 2.2.** Let $1 \leq p < \infty$ and $\alpha$ be a multi-index. Suppose $f$ is harmonic on a proper open subset $\Omega$ of $\mathbb{R}^n$. Then, we have

$$|\partial^\alpha f(x)|^p \leq \frac{C}{d^{n+|\alpha|}(x, \partial \Omega)} \int_\Omega |f(y)|^p \, dy \quad (x \in \Omega)$$

where $d(x, \partial \Omega)$ denotes the distance from $x$ to $\partial \Omega$. The constant $C$ depends only on $n$, $p$, and $\alpha$.

We are now ready to prove the following size estimates of derivatives of the reproducing kernel.

**Theorem 2.3.** Given multi-indices $\alpha$ and $\beta$, there exists a positive constant $C = C(\alpha, \beta)$ such that

$$|\partial_x^\alpha \partial_y^\beta R(x, y)| \leq \frac{C}{\rho^{n+|\alpha|+|\beta|}}(x, y)$$

for all $x, y \in B$.

**Proof.** Note that

$$\int_B |R(x, y)|^2 \, dy = R(x, x) \approx \frac{1}{(1 - |x|)^n}$$

for all $x \in B$. Thus, given a multi-index $\beta$, we have

$$|\partial_y^\beta R(x, y)|^2 \lesssim \frac{1}{(1 - |x|)^n (1 - |y|)^{n+2|\beta|}}$$

for all $x, y \in B$ by Lemma 2.2. Given another multi-index $\alpha$, applying a standard argument using Cauchy’s estimate, we conclude from the above

$$|\partial_x^\alpha \partial_y^\beta R(x, y)| \lesssim \frac{1}{(1 - |x|)^{n/2+|\alpha|}(1 - |y|)^{n/2+|\beta|}}$$
for all \(x, y \in B\). Thus, for the case where \(|x - y| < (1 - |x|)/2\) or \(|x - y| < (1 - |y|)/2\), we have \(\rho(x, y) \lesssim 1 - |x| \approx 1 - |y|\) by (2.3) and therefore obtain the desired estimate from (2.5).

Now, assume \(|x - y| \geq (1 - |x|)/2\) and \(|x - y| \geq (1 - |y|)/2\). Note

\[
R(x, y) = C_1 \eta(x, y) \rho^{-(n+2)}(x, y) + C_2 |x|^2 |y|^2 \rho^{-n}(x, y)
\]

where \(\eta(x, y) = (1 - |x|^2|y|^2)^2\). Let \(\eta = \eta(x, y)\) and \(\rho = \rho(x, y)\). Here, we will estimate the first term. The estimate of the second term is similar and simpler. By Lemma 2.1 we have

\[
|\partial_x^\alpha \partial_y^\beta (\eta \rho^{-n-2})| \leq \sum |\partial_x^\alpha \partial_y^\beta \eta| |\partial_x^\alpha \partial_y^\beta \rho^{-n-2}| \lesssim 1 \lesssim 1 \lesssim \rho^{2|\alpha'| + |\beta'| + 1 - |\alpha| - |\beta|},
\]

where the sum is taken over all \(\alpha', \alpha'', \beta', \beta''\) such that \(|\alpha'| + |\alpha''| = |\alpha|\) and \(|\beta'| + |\beta''| = |\beta|\). For \(|\alpha'| + |\beta'| \geq 2\), we have

\[
|\partial_x^\alpha \partial_y^\beta \eta| \lesssim 1 \lesssim \rho^{2|\alpha'| + |\beta'| + 1 - |\alpha| - |\beta|},
\]

Note \(\nabla_x \eta = -4x|\eta|^2(1 - |x|^2|y|^2)\) and \(\nabla_y \eta = -4y|\eta|^2(1 - |x|^2|y|^2)\). Thus,

\[
|\nabla_x \eta| + |\nabla_y \eta| \lesssim 1 - |x||y| = 1 - |x| + |x|(1 - |y|) \lesssim |x - y| \leq \rho
\]

so that (2.7) remains valid for all \(\alpha', \beta'\). Therefore, by (2.6) and (2.7), we have

\[
|\partial_x^\alpha \partial_y^\beta (\eta \rho^{-n-2})| \lesssim \rho^{-n-|\alpha'| + |\beta'| + 1 - |\alpha| - |\beta|} \approx \rho^{-n-|\alpha| - |\beta|},
\]

which completes the proof.

In conjunction with Theorem 2.3, the following two propositions describe integral behavior of derivatives of the reproducing kernel. Here and in the rest of the paper, we put

\[
\delta(x) = 1 - |x|^2 \quad (x \in B)
\]

for simplicity. Also, we use the notation \(d\sigma\) for the surface area measure on \(\partial B\).

**Proposition 2.4.** Given \(c\) real, define

\[
I_c(x) = \int_B \frac{dy}{\rho^{n+c}(x, y)}
\]

for \(x \in B\). Then the following hold.

1. For \(c < 0\), \(I_c\) is bounded on \(B\).
2. For \(c = 0\), \((1 + \log \delta^{-1})^{-1} I_0\) is bounded on \(B\).
3. For \(c > 0\), \(\delta^c I_c\) is bounded on \(B\).

**Proof.** Here, we give a proof for \(c \geq 0\) (which are the cases we need later). The case \(c < 0\) is easily modified and left to the readers. Assume \(c \geq 0\). Recall that the integral of the Poisson kernel on \(\partial B\) is constant. That is,

\[
\int_{\partial B} \frac{1 - |\zeta|^2}{|y - \zeta|^n} d\sigma(\zeta) = \sigma(\partial B)
\]
for all \( y \in B \). Hence, integrating in polar coordinates, we have
\[
I_s(x) = \int_0^1 t^{n-1} \int_{\partial B} \frac{d\sigma(\zeta)}{(1 - 2tx \cdot \zeta + t^2 |x|^{2(n+\epsilon)})^{\frac{1}{2}}}
\]
so that
\[
I_s(x) \leq \int_0^1 t^{n-1} \int_{\partial B} \frac{d\sigma(\zeta)}{|x - \zeta|^{n+\epsilon}} dt,
\]
for \( x \in B \). The rest of the proof is now straightforward. \qed

**Proposition 2.5.** Given \( s > 1 \), there exists a constant \( C = C(s) \) such that
\[
\int_0^1 \frac{dt}{\rho^s(tx, y)} \leq \frac{C}{\rho^{s-1}(x, y)}
\]
for all \( x, y \in B \).

**Proof.** Fix \( x, y \in B \). Note that if \( x \cdot y \leq 0 \) or \( |x||y| < 1/2 \), then all the terms are bounded above and bounded away from 0. Thus the estimate is trivial. So suppose \( x \cdot y > 0 \) and \( |x||y| \geq 1/2 \) in the rest of the proof. Define \( h(t) = \rho(tx, y) \) for \( t \geq 0 \) and put \( t_0 = x \cdot y / |x|^2 |y|^2 > 0 \). Then the function \( h \) attains its minimum at \( t_0 \) with minimum value \( \sqrt{1 - (x \cdot y)^2 / |x|^2 |y|^2} \).

Note that if \( t_0 < 1 \), then \( (x \cdot y)^2 / |x|^2 |y|^2 < x \cdot y \) and so
\[
2h^2(t_0) - h^2(1) \geq 2(1 - x \cdot y) - (1 - 2x \cdot y + |x|^2 |y|^2) = 1 - |x|^2 |y|^2 \geq 0.
\]
Thus, \( h(t_0) \leq h(1)/\sqrt{2} \). Now, since \( |x||y| \geq 1/2 \), we have
\[
h^2(t) = |x|^2 |y|^2 (t - t_0)^2 + h^2(t_0) \geq (t - t_0)^2 + h^2(1)
\]
and thus
\[
\int_0^1 \frac{dt}{h^s(t)} \leq \int_0^1 \frac{dt}{((t - t_0)^2 + h^2(1))^{s/2}} \leq \int_{-\infty}^\infty \frac{dt}{(t^2 + h^2(1))^{s/2}} \approx \frac{1}{h^{s-1}(1)} \int_{-\infty}^\infty \frac{dt}{(t^2 + 1)^{s/2}}.
\]
Note that the last integral is finite for \( s > 1 \).

Note also that if \( t_0 \geq 1 \), then it is elementary to see
\[
2h^2(t) \geq |x|^2 |y|^2 (t - 1)^2 + h^2(1) \geq (t - 1)^2 + h^2(1)
\]
for all \( t \in [0, 1] \), because \( |x||y| \geq 1/2 \). Hence, a similar argument yields the desired estimate. \qed
3. Reproducing Formulas

By the reproducing formula (2.1), the orthogonal projection \( Q : L^2 \to b^2 \) is given by
\[
Qg(x) = \int_B g(y) R(x, y) \, dy \quad (x \in B)
\]
for \( g \in L^2 \). Note that \( Q \) naturally extends to an operator from \( L^1 \) into the space of all harmonic functions on \( B \). In this section, as a preliminary step towards our results in later sections, we first prove that there are many other (nonorthogonal) projections which can be easily estimated by means of results in the previous section.

**Lemma 3.1.** Given a positive integer \( m \), there are constants \( c_j = c_j(m) \) with \( c_m = \frac{(-1)^{m+1}}{m!} \), and \( c_{jk} = c_{jk}(m) \) such that
\[
\int_B \psi \Delta^m (\delta^{2m} \varphi) \, dV = \int_B [\Delta \psi][\Delta^{m-1}(\delta^{2m} \varphi)] \, dV + \int_{\partial B} \psi \Delta^{m-1}(\delta^{2m} \varphi) \, d\sigma - \int_{\partial B} [\Delta \psi][\Delta^{m-1}(\delta^{2m} \varphi)] \, d\sigma.
\]
whenever \( \psi \in b^1 \) and \( \varphi \) is a function harmonic on an open set containing \( \overline{B} \).

**Proof.** Suppose \( \varphi \) and \( \psi \) are functions harmonic on an open set containing \( \overline{B} \). By Green’s theorem, we have
\[
\int_B \psi \Delta^m (\delta^{2m} \varphi) \, dV = \int_B [\Delta \psi][\Delta^{m-1}(\delta^{2m} \varphi)] \, dV + \int_{\partial B} \psi \Delta^{m-1}(\delta^{2m} \varphi) \, d\sigma - \int_{\partial B} [\Delta \psi][\Delta^{m-1}(\delta^{2m} \varphi)] \, d\sigma.
\]
Clearly, the first term of the right side of the above is 0. One can check that the remaining terms are also 0, because \( \Delta \Delta^{m-1}(\delta^{2m} \varphi) \) and \( \Delta^{m-1}(\delta^{2m} \varphi) \) both vanish on \( \partial B \). It follows that
\[
\int_B \psi \Delta^m (\delta^{2m} \varphi) \, dV = 0.
\]
Note that, since harmonic polynomials are dense in \( b^1 \), this remains valid for general \( \psi \in b^1 \).

We now calculate \( \Delta^m (\delta^{2m} \varphi) \). For an integer \( k \geq 2 \), a straightforward calculation shows that \( \nabla \delta^k (x) = -2k \delta^{k-1}(x) \) and
\[
\Delta \delta^k = -2k(n + 2k - 2)\delta^{k-1} + 4k(k - 1)\delta^{k-2}.
\]
Hence, we obtain
\[
\Delta (\delta^k \varphi) = \Delta (\delta^k \varphi) + 2 \nabla \delta^k \cdot \nabla \varphi = 4k(k - 1)\delta^{k-2} \varphi + 4k \delta^{k-1} \Delta \varphi - 2k(n + 2k - 2)\delta^{k-1} \varphi.
\]
Let \( k = 2m \) and apply the Laplacian \( m \)-times using this formula recursively. Since radial derivatives of harmonic functions are again harmonic, the result is
\[
\Delta^m (\delta^{2m} \varphi) = 4^m (2m)! \varphi + \sum_{j=1}^m c_j^j \delta^j \Delta^j \varphi + \sum_{j=1}^m \sum_{k=0}^{m-1} c_{jk}^j \delta^j \Delta^k \varphi,
\]
for some constants \( c_j^j \) and \( c_{jk}^j \) depending only on \( n \) and \( m \). Note that \( c_m^m = (-4)^m \frac{(2m)!}{m!} \). This, together with (3.1), proves the lemma. \( \square \)
For a given positive integer \( m \), let \( c_j \) and \( c_{jk} \) be the constants provided by Lemma 3.1 and define an operator \( T_m \) by

\[
T_m g = \sum_{j=1}^{m} c_j \delta^j g + \sum_{j=1}^{m-1} \sum_{k=0}^{j-1} c_{jk} \delta^{j+k} D^k g
\]

for \( g \in C^m(B) \). For \( x \in B \), let \( R_x = R(x, \cdot) \), put \( R_m(x, y) = T_m R_x(y) \), and consider an operator \( Q_m \) defined by

\[
Q_m g(x) = \int_B g(y) R_m(x, y) \, dy \quad (x \in B)
\]

for \( g \in L^1 \). Note that \( Q_m \) is a linear operator taking \( L^1 \) into the space of all harmonic functions on \( B \). It is well known that \( Q \) is bounded on \( L^p \) if and only if \( 1 < p < \infty \). The advantage of \( Q_m \) is the boundedness on \( L^p \) for all \( 1 \leq p < \infty \).

**Theorem 3.2.** Let \( 1 \leq p < \infty \) and \( m \) be a positive integer. Then \( Q_m : L^p \to \mathfrak{b}^p \) is bounded. Moreover, \( Q_m f = f \) for \( f \in \mathfrak{b}^p \).

**Proof.** Given \( f \in \mathfrak{b}^p \) and \( x \in B \), apply Lemma 3.1 with \( \psi = f \) and \( \varphi = R(x, \cdot) \). Then, by the reproducing formula (2.1), we obtain \( Q_m f = f \).

Now, we show the \( L^p \)-boundedness of \( Q_m \). Let \( \psi \in L^p \). Then \( |Q_m \psi| \leq \sum_{j=0}^{m} |\Lambda_j \psi| \) where

\[
\Lambda_0 \psi(x) = \int_B |\psi(y)| \delta(y) \, dy
\]

\[
\Lambda_j \psi(x) = \int_B |\psi(y)| \delta^j(y) \, dy \quad (j = 1, \ldots, m)
\]

for \( x \in B \). First, consider \( p = 1 \). By (2.2) and Proposition 2.4, we have

\[
\int_B \Lambda_0 \psi(x) \, dx \leq \int_B \delta(y) |\psi(y)| \int_B |R(x, y)| \, dy \, dy \lesssim \int_B |\psi(y)| \, dy.
\]

A similar argument using Theorem 2.3 and Proposition 2.4 yields \( L^1 \)-boundedness of each \( \Lambda_j \). Now, assume \( 1 < p < \infty \). By Theorem 2.3, we have

\[
\Lambda_j \psi(x) \lesssim \int_B \frac{|\psi(y)| \delta^j(y)}{\rho^{n+1}(x, y)} \, dy \lesssim \int_B \frac{|\psi(y)|}{\rho^n(x, y)} \, dy
\]

for all \( x \in B \) and \( j = 0, 1, \ldots, m \). Hence, by the Schur test as in [5], each \( \Lambda_j \) is bounded on \( L^p \). \( \square \)

As another consequence of Lemma 3.1, we have the following reproducing formula for harmonic Bloch functions. This reproducing formula will play an essential role in the estimates of harmonic Bloch functions in the next section. For \( m = 1 \), one can find in [5] another proof by means of the extended Poisson kernel.

**Theorem 3.3.** Let \( m \) be a positive integer. Then \( T_m : \mathcal{B} \to \mathcal{B}^\infty \) is bounded. Moreover, \( T_m \) takes \( \mathcal{B}_0 \) into \( \mathcal{C}_0 \) and \( QT_m f = f \) for \( f \in \mathcal{B} \).

It is well known (and not hard to prove) that \( Q : \mathcal{B}^\infty \to \mathcal{B} \) bounded and \( Q(\mathcal{C}_0) \subset \mathcal{B}_0 \). Note that the above theorem yields \( Q(\mathcal{B}^\infty) = \mathcal{B} \) and \( Q(\mathcal{C}_0) = \mathcal{B}_0 \), which are also well known.
Proof. Let \( f \in B \). For \( m = 1 \), we have \( T_1 f = \delta D f + (n/2 + 1)\delta f \). It is not hard to see that

\[
|f| \lesssim \|f\|_B (1 + \log \delta^{-1}).
\]

So, \( T_1 f \in L^\infty \) and \( \|T_1 f\|_\infty \lesssim \|f\|_B \). Now, given \( x \in B \), apply Lemma 3.1 with \( \psi = R(x, \cdot) \) and \( \varphi = f_r \), \( 0 < r < 1 \), where \( f_r \) is a dilate defined by \( f_r(y) = f(ry) \) for \( y \in B \). Then, by the reproducing formula (2.1), we obtain \( QT_1 f_r = f_r \). Note \( D f_r = (D f)_r \rightarrow D f \) and \( \|D f_r\| \leq \|D f\|_\infty \). Thus, \( \delta D f_r \rightarrow \delta D f \) in \( L^1 \) by the Lebesgue dominated convergence theorem. Similarly, \( \delta f_r \rightarrow \delta f \) in \( L^1 \). Therefore, after taking the limit, we have \( QT_1 f = f \), or more explicitly,

\[
f(x) = \int_B T_1 f(y) R(x, y) \, dy \quad (x \in B)
\]

and thus, differentiating under the integral, we have

\[
\partial^\alpha f(x) = \int_B T_1 f(y) \partial^\alpha_x R(x, y) \, dy
\]

for every multi-index \( \alpha \) and \( x \in B \). Thus, for \( |\alpha| \geq 1 \), we obtain by Theorem 2.3 and Proposition 2.4

\[
\delta^{[\alpha]}(x) \partial^\alpha f(x) \lesssim \|f\|_B \delta^{[\alpha]}(x) \int_B \frac{dy}{\rho_n^{\alpha+1}[x, y]} \lesssim \|f\|_B
\]

for all \( x \). It follows that \( \|\delta^k D^k f\|_\infty \lesssim \|f\|_B \) for every positive integer \( k \) and therefore \( \|T_m f\|_\infty \lesssim \|f\|_B \). In particular, the integral \( QT_m f \) is well defined. One can check \( D^k f_r = (D^k f)_r \), for each \( k \). Thus, by the same limiting argument, we have \( QT_m f = f \) for general \( m \).

Now, we prove \( T_m (B_0) \subset C_0 \). Suppose \( f \in B_0 \). Since \( f \in B_0 \), we have \( \delta D f \in C_0 \). Also, \( \delta f \in C_0 \) by (3.2). Thus, \( T_1 f \in C_0 \). Note that, for each \( 0 < r < 1 \) and \( \alpha \) with \( |\alpha| \geq 1 \), we have by Proposition 2.4 and (3.3)

\[
\delta^{[\alpha]}(x) \partial^\alpha f(x) \\
\lesssim \delta^{[\alpha]}(x) \int_{|y| > r} \frac{|T_1 f(y)|}{\rho_n^{\alpha+1}[x, y]} \, dy + \delta^{[\alpha]}(x) \int_{|y| \leq r} \frac{|T_1 f(y)|}{\rho_n^{\alpha+1}[x, y]} \, dy \\
\lesssim \sup_{|y| > r} |T_1 f(y)| + \|f\|_B \delta^{[\alpha]}(x) \int_{|y| \leq r} \frac{dy}{\rho_n^{\alpha+1}[x, y]}.
\]

Now, take the limit \( |x| \to 1 \) with \( r \) fixed and get

\[
\limsup_{|x| \to 1} \delta^{[\alpha]}(x) \partial^\alpha f(x) \lesssim \sup_{|y| > r} |T_1 f(y)|.
\]

Since \( r \) is arbitrary and \( T_1 f \in C_0 \), we obtain \( \delta^j D^j f \in C_0 \) for \( |\alpha| \geq 1 \) and hence \( \delta^j D^j f \in C_0 \) for each \( j \geq 1 \). Consequently, since \( |T_m f| \lesssim |\delta f| + \sum_{j=1}^m |\delta^j D^j f| \), we conclude \( T_m f \in C_0 \).

\[
4. \text{ Gleason’s Problem}
\]

Let \( f \in C^1(B) \). Then, for \( x \in B \), we have

\[
f(x) - f(0) = \int_0^1 x \cdot \nabla f(tx) \, dt = \sum_{j=1}^n x_j A_j f(x)
\]
where
\[ A_j f(x) = \int_0^1 \partial_j f(tx) \, dt. \]

Note that if \( f \) is harmonic, then so is each \( A_j f \). Gleason’s problem is to figure out whether the operators \( A_j \) leave the spaces under consideration invariant. Recently the same problem has been considered on the setting of the upper half-space by Choe-Koo-Yi [2]. Their result reveals some pathology caused by unboundedness of the half-space. Also, see [8] for more references in this direction. For the ball, operators \( A_j \) are expected to behave well, which turns out to be indeed the case. Based on all the estimates in the previous sections, proofs are also quite simple.

**Theorem 4.1.** Let \( 1 \leq p < \infty \). The operators \( A_j \) are all bounded on \( b^p \).

**Proof.** First, consider the case \( 1 < p < \infty \). Let \( f \in b^p \). Differentiating under the integral sign of the reproducing formula (2.1), we have
\[ \partial_j f(x) = \int_B f(y) \frac{\partial R}{\partial x_j}(x, y) \, dy \]
and thus by Theorem 2.3
\[ |\partial_j f(x)| \leq \int_B |f(y)| \left| \frac{\partial R}{\partial x_j}(x, y) \right| \, dy \leq \int_B \frac{|f(y)|}{\rho^{n+1}(x, y)} \, dy \]
for \( x \in B \). It follows from Proposition 2.5 that
\[ |A_j f(x)| \lesssim \int_B |f(y)| \left( \frac{1}{\rho^n(x, y)} + \frac{1}{\rho^{n+1}(x, y)} \right) \, dy \]
for each \( j \) and \( x \in B \). Thus, by the Schur test as in [5], we get the boundedness of \( A_j \) on \( b^p \).

Next, consider the case \( p = 1 \) and let \( f \in b^1 \). This time we use the reproducing formula given by Theorem 3.2 with \( m = 1 \). Then, by a similar argument using Proposition 2.5 and Theorem 2.3, we obtain
\[ |A_j f(x)| \lesssim \int_B \delta(y) |f(y)| \left( \frac{1}{\rho^n(x, y)} + \frac{1}{\rho^{n+1}(x, y)} \right) \, dy \]
for all \( j \) and \( x \in B \). Integrating both sides of the above, we obtain from Proposition 2.4 that
\[ \int_B |A_j f(x)| \, dx \lesssim \int_B \delta(y) |f(y)| \int_B \left( \frac{1}{\rho^n(x, y)} + \frac{1}{\rho^{n+1}(x, y)} \right) \, dx \, dy \]
\[ \lesssim \int_B |f(y)| \, dy \]
as desired. \( \square \)

The analogous result is valid for the harmonic Bloch space.

**Theorem 4.2.** The operators \( A_j \) are all bounded on \( B \). In addition, Each \( A_j \) takes \( B_0 \) into itself.

**Proof.** Now suppose \( f \in B \) and let \( x \in B \). Then, by Theorem 3.3, we have
\[ f(x) = \int_B T_1 f(y) R(x, y) \, dy. \]
Now, differentiating under the integral, we have

\[ A_j f(x) = \int_0^1 \int_B T_1 f(y) \frac{\partial R}{\partial x_j}(tx, y) dy \, dt \]

for each \( j \). Recall \( |\nabla T_1 f| \lesssim \|f\| \). It follows from Theorem 2.3 and Proposition 2.5 that

\[ |\nabla A_j f(x)| \lesssim \|f\| \int_0^1 \int_B \frac{t}{\rho^{n+2}(tx, y)} dy \, dt \lesssim \|f\| \int_B \frac{dy}{\rho^{n+1}(x, y)} \lesssim \|f\| \rho(1 - |x|^2)^{-1}. \]

In other words, \( \|A_j f\| \lesssim \|f\| \). For \( f \in \mathcal{B}_n \), we have by Proposition 2.4

\[ |\nabla A_j f(x)| \lesssim \int_{|x| > r} \frac{|T_1 f(y)|}{\rho^{n+1}(x, y)} dy + \int_{|x| \leq r} \frac{|T_1 f(y)|}{\rho^{n+1}(x, y)} dy \]

for \( 0 < r < 1 \). Thus, the same argument as in the proof of Theorem 3.3 gives \( \partial \nabla A_j f \in C_0 \) so that \( A_j f \in \mathcal{B}_0 \).

Now, repeating the results of Theorem 4.1 and Theorem 4.2, we can prove Theorem 1.2.

**Proof of Theorem 1.2**: For \( f \) harmonic on \( B \), a repetition of (4.1) yields

\[ f(x) = f(0) + \sum_{j=1}^n x_j A_j f(x) = f(0) + \sum_{j=1}^n A_j f(0) x_j + \sum_{j=1}^n \sum_{k=1}^n x_j x_k A_k A_j f(x) = f(0) + \nabla f(0) \cdot x + \sum_{j=1}^n \sum_{k=1}^n x_j x_k A_k A_j f(x) \]

for \( x \in B \). The boundedness properties of operators \( A_k A_j \) follow from Theorem 4.1 and Theorem 4.2. For higher orders, one may repeat the same argument. \( \square \)

### 5. Derivative Norms

In this section we prove the equivalence of various derivative norms. We will consider radial, tangential and ordinary derivative norms. For the half-space, such results are proved in [3]. For the holomorphic Bergman spaces on the ball, such results (only for ordinary derivative norms) are proved in [8]. Our approach is direct and quite different from theirs.

Since there is no smooth nonvanishing tangential vector field on \( \partial B_n \) for \( n > 2 \), we define tangential derivatives by means of a family of tangential vector fields generating all the tangent vectors. We define \( T^\alpha f \) of \( f \in C^1(B) \) by

\[ T^\alpha f(x) = (x_i \partial_j - x_j \partial_i) f(x) \quad (x \in B) \]

for \( 1 \leq i < j \leq n \). Note that tangential derivatives of harmonic functions are again harmonic. Given a nontrivial multi-index \( \alpha \), we abuse the notation \( T^\alpha = T_{i_1 j_1} \cdots T_{i_n j_n} \) for any choice of \( i_1, \ldots, i_n \) and \( j_1, \ldots, j_n \).
Now, we introduce corresponding norms. For $1 \leq p < \infty$ and positive integers $m$, put

$$
\|f\|_{p,m,1} = |f(0)| + \|\delta^m D^m f\|_p \\
\|f\|_{p,m,2} = |f(0)| + \sum_{|\alpha| = m} \|\delta^m \partial^\alpha f\|_p \\
\|f\|_{p,m,3} = \sum_{|\alpha| < m} |\partial^\alpha f(0)| + \sum_{|\alpha| = m} \|\delta^m \partial^\alpha f\|_p
$$

for functions $f$ harmonic on $B$. Our result is that all of these norms are equivalent. Estimates are somehow long and thus we proceed step by step through lemmas.

**Lemma 5.1.** Given an integer $m \geq 1$, there exists a smooth differential operator $E_m$ of order $2m - 1$ with bounded coefficients such that

$$
D^{2m} f = \left( -\sum_{i<j} T_{ij}^2 \right)^m f + E_m f
$$

for functions $f$ harmonic on $B$.

**Proof.** Since $D^2 = \sum x_i x_j \partial_i \partial_j + K_1$, we have

$$
\sum_{i<j} T_{ij}^2 = \sum_{i<j} (x_i^2 \partial_j^2 + x_j^2 \partial_i^2 - 2x_i x_j \partial_i \partial_j) + K_2
$$

$$
= \sum_{i\neq j} (x_i^2 \partial_j^2 - x_i x_j \partial_i \partial_j) + K_2
$$

$$
= \sum_{i,j} (x_i^2 \partial_j^2 - x_i x_j \partial_i \partial_j) + K_2
$$

$$
= |x|^2 \Delta - D^2 + K_1 + K_2.
$$

Here, $K_1$ and $K_2$ are first order smooth differential operators with bounded coefficients. Accordingly,

$$
\left( \sum_{i<j} T_{ij}^2 \right)^m = (-1)^m D^{2m} + K_3 \Delta + K_4
$$

for some $K_3$ of order $2m - 2$ and $K_4$ of order $2m - 1$. This implies the lemma. \qed

The following is an easy consequence of the mean value property of $\nabla f$:

$$
(5.1) \quad \sup_{|x| \leq r} |f(x) - f(0)|^p \leq C \int_{|x| \leq r+\epsilon} |\nabla f(x)|^p \, dx
$$

for $1 \leq p < \infty$, $\epsilon > 0$, $0 < r < 1 - \epsilon$, and functions $f$ harmonic on $B$. The constant $C$ is independent of $f$ and $r$. The analogous inequalities for radial and tangential derivatives do not seem to be trivial. Recall that radial and tangential derivatives of harmonic functions are again harmonic.

**Proposition 5.2.** Let $1 \leq p < \infty$, $\epsilon > 0$ and $m$ be a positive integer. Then there exists a constant $C = C(p,m,\epsilon)$ such that

$$
(1) \quad \sup_{|x| \leq r} |f(x) - f(0)|^p \leq C \int_{|x| \leq r+\epsilon} |D^m f(x)|^p \, dx
$$
(2) \( \sup_{|x| \leq r} |f(x) - f(0)|^p \leq C \sum_{j=1}^{n} \int_{|x| \leq r + \epsilon} |T^o f(x)|^p \ dx \)

whenever \( 0 < r < 1 - \epsilon \) and \( f \) is harmonic on \( B \).

**Proof.** Suppose \( 0 < r < 1 - \epsilon \) and let \( f \) be a function harmonic on \( B \). By Lemma 2.2,

\[
\sup_{|x| \leq r} |f(x) - f(0)|^p \lesssim \int_{|x| \leq r + \epsilon/2} |f(x) - f(0)|^p \ dx \\
\lesssim e^{-n} \int_{|x| \leq r + \epsilon/2} |Df(x)|^p \ dx \\
\lesssim \sup_{|x| \leq r + \epsilon/2} |Df(x)|^p
\]

where the second inequality is easily verified by using the homogeneous expansion of \( f \) (see Theorem 5.3 of [1]). Thus,

\[
\sup_{|x| \leq r} |f(x) - f(0)|^p \lesssim \sup_{|x| \leq r + \epsilon/2} |Df(x)|^p \lesssim \int_{|x| \leq r + \epsilon} |Df(x)|^p \ dx
\]

This proves the lemma for \( m = 1 \). For general \( m \), one can repeat the same process with harmonic functions \( D^j f \). Since \( D^j f(0) = 0 \) for \( j \geq 1 \), we get (1).

We now prove (2). Let \( S_r \) denote the sphere of radius \( r \) centered at the origin. Now, fix \( x \in S_r \), \( \xi \in \partial B \) and pick a smooth curve \( \gamma: [0, 1] \to S_r \) such that \( \gamma(0) = x \), \( \gamma(t_0) = r\xi \) for some \( t_0 \) and \( |\gamma'| = 2\pi r \). We claim

\[
[(f \circ \gamma)'(t)] \leq \sum_{i<j} |T_{ij} f(\gamma(t))| \tag{5.2}
\]

for all \( t \in [0, 1] \). Given \( t \in [0, 1] \), let \( \gamma(t) = r \xi \) where \( \xi \in \partial B \). Since \( |\xi| = 1 \), we may assume \( |\xi_j| \geq \frac{1}{\sqrt{n}} \).

Let \( e_{1j} = \xi_1 e_j - \xi_j e_1, \) \( 2 \leq j \leq n \), where \( e_k \) denotes the unit vector in the positive direction of the \( k \)-th coordinate axis. Note that \( \{e_{1j}\}_{j \geq 2} \) is a basis for the tangent space \( T_\xi \) to \( \partial B \) at \( \xi \). Following the Gram-Schmidt process, put \( a_2 = e_{12} \) and define

\[
a_j = e_{1j} - \sum_{k=2}^{j-1} \frac{a_k \cdot e_{1j}}{a_k} a_k \tag{5.3}
\]

inductively for \( j = 3, ..., n \). Then \( \{a_j\}_{j \geq 2} \) is an orthogonal basis for \( T_\xi \). What we need here is a uniform control of coefficients. Since each \( a_k \) in the sum of the above is spanned by \( e_{12}, ..., e_{1k} \) by construction, the \( j \)-th coordinate of \( a_j \) comes only from \( e_{1j} \) and is equal to \( \zeta_j \). Thus, \( |a_j| \geq 1/\sqrt{n} \) for each \( j \). Now, writing (5.3) in the form \( a_j = \sum_{k=0}^{j} c_{jk} e_{1k} \), one may check inductively \( |c_{jk}| < C \) for some \( C \) which depends only on \( n \). Now, since \( \gamma'(t) \in T_\xi \), we have

\[
\gamma'(t) = \sum_{j=2}^{n} \frac{d_j}{|a_j|} \sum_{k=2}^{j} c_{jk} e_{1k} = \sum_{k=2}^{n} \sum_{j=k}^{n} \frac{d_{jk}}{|a_j|} e_{1k}
\]

for some constants \( d_j \) such that \( \sum_{j=k}^{n} d_{jk}^2 = |\gamma'(t)|^2 = (2\pi r)^2 \). Since \( \nabla f(\gamma(t)) \cdot e_{1k} = r^{-1} T_{1k} f(\gamma(t)) \), it follows that

\[
|\nabla f(\gamma(t)) \cdot \gamma'(t)| \leq \sum_{k=2}^{n} \sum_{j=k}^{n} \frac{|d_j||c_{jk}|}{r|a_j|} |T_{1k} f(\gamma(t))| \lesssim \sum_{k=2}^{n} |T_{1k} f(\gamma(t))|,
\]
which implies (5.2). Now, by (5.2), we have
\[ |f(x) - f(r\zeta)|^p \leq \int_0^a |(f \circ \gamma')(t)|^p \, dt \lesssim \sup_{S_r} \sum_{i<j} |T_{ij}f|^p \]
and therefore
\[ |f(x) - f(0)|^p \leq \int_{\partial B} |f(x) - f(r\zeta)|^p \, d\sigma(\zeta) \lesssim \sup_{S_r} \sum_{i<j} |T_{ij}f|^p \]
by subharmonicity. On the other hand, by Lemma 2.2
\[ \sup_{S_r} |T_{ij}f|^p \lesssim e^{-n} \int_{|y|\leq n} |T_{ij}f(y)|^p \, dy \]
for all $T_{ij}$. This proves the lemma for $m = 1$ by the maximum principle. For general $m$, one can repeat the same process with harmonic functions $T^0f$. Since $T^0f(0) = 0$ for $|x| \geq 1$, we obtain (2).

Lemma 5.3. For $p \geq 1$ and $r > -1$, we have
\[ \int_0^a |h(t)|^p \, dt \leq \left( \frac{p}{r+1} \right)^p \left\{ \int_0^a |h'(t)|^p t^r \, dt + |h(a)|^p \right\} \]
for $C^1$-functions $h$ on $(0, a]$, $a > 0$.

Proof. Suppose $h$ is a $C^1$-function on $(0, a]$. Then we have
\[ |h(t)| \leq |h(a)| + \int_t^a |h'(s)| \, ds \]
for $0 < t \leq a$. Thus, the inequality follows from Hardy’s inequality (see, for example, [6]).

We are now ready to prove norm equivalence for the harmonic Bergman spaces.

Theorem 5.4. Let $1 \leq p < \infty$ and $m$ be a positive integer. Then, there are positive constants $C_1, C_2, C_3, C_4$ such that
\[ \|f\|_p \leq C_1\|f\|_{p,m,1} \leq C_2\|f\|_{p,m,2} \leq C_3\|f\|_{p,m,3} \leq C_4\|f\|_p \]
for functions $f$ harmonic on $B$.

In the proof below $f$ is a given harmonic function on $B$.

Proof of $\|f\|_{p,m,3} \lesssim \|f\|_p$: Given $x \in B$, take $\Omega$ to be the ball $B_x$ with center at $x$ and radius $(1 - |x|)/2$. Note $(1 - |x|)/2 \leq 1 - |y| \leq 3(1 - |x|)/2$ for $y \in B_x$. Hence, for a multi-index $\alpha$ and $r$ real, we obtain by Lemma 2.2
\[ (1 - |x|)^{|\alpha|+r} |\partial^\alpha f(x)|^p \lesssim \frac{1}{(1 - |x|)^n} \int_{B_x} |f(y)|^p (1 - |y|)^r \, dy. \]
(5.4)

Inserting $x = 0$ into the above, we have
\[ \sum_{|\alpha|<m} |\partial^\alpha f(0)|^p \lesssim \int_B |f|^p \delta^r \, dy. \]
Also, for $|x| = m$, we obtain by (5.4)

$$\sum_{|r| = m} \int_B |f|^{p+r} \delta^r dV \lesssim \int_B (1-|x|)^n \int_{B_r} |f(y)|^{p+r} dy dx$$

$$= \int_B |f(y)|^{p+r} \int_B \frac{X_{B_r}(y)}{(1-|x|)^n} dx dy$$

(5.5)

Consequently, taking $r = 0$, we have $\|f\|_{p,m,0} \lesssim \|f\|_p$. $\square$

**Proof of $\|f\|_{p} \lesssim \|f\|_{p,m,1}$**: For a given $\zeta \in \partial B$ and $r > -1$, apply Lemma 5.3 to the function $h(t) = f((1-t)\zeta)$ on $(0, 1/2]$. What we get is

$$\int_{1/2}^1 |f(t\zeta)|^{p(1-t)^r} dt \leq \int_{1/2}^1 \zeta \cdot \nabla f(t\zeta)|^{p(1-t)^{p+r}} dt + |f(\zeta/2)|^p$$

and thus

$$\int_{1/2}^1 |f(t\zeta)|^{p(1-t)^r t^{n-1}} dt \leq \int_{1/2}^1 |D f(t\zeta)|^{p(1-t^2)^{p+r}} dt + \sup_{|x| = 1/2} |f(x)|^p.$$

Integrating both sides of the above on $\partial B$, we have

$$\int_{|x| > 1/2} |f|^{p+r} dV \lesssim \int_{|x| 

Note that

$$\sup_{|x| \leq 1/2} |f(x)|^p \lesssim |f(0)|^p + \int_{|x| \leq 3/4} |D f(x)|^p dx \lesssim |f(0)|^p + \int_B |D f|^p dV$$

by Proposition 5.2. It follows that

$$\int_B |f|^{p+r} dV \lesssim \int_B |D f|^{p+r} dV + |f(0)|^p.$$

Note $D f(0) = 0$. Thus, iterating the above with harmonic functions $D^i f$, we have

$$\int_B |f|^{p+r} dV \lesssim \int_B |D^n f|^{p+r} dV + |f(0)|^p.$$

So, taking $r = 0$, we have $\|f\|_p \lesssim \|f\|_{p,m,1}$. $\square$

**Proof of $\|f\|_{p,m,1} \lesssim \|f\|_{p,m,2}$**: This time we apply Lemma 5.3 on the interval $(0, 1]$. We obtain, for a given $\zeta \in \partial B$ and $r > -1$,

$$\int_0^1 |f(t\zeta)|^{p(1-t)^r} dt \leq \int_0^1 \zeta \cdot \nabla f(t\zeta)|^{p(1-t)^{p+r}} dt + |f(0)|^p$$

and thus

$$\int_0^1 |f(t\zeta)|^{p(1-t^2)^{p+r}} dt \lesssim \int_{1/2}^1 |\nabla f(t\zeta)|^{p(1-t^2)^{p+r}} dt + \sup_{|x| \leq 1/2} |\nabla f(x)|^p + |f(0)|^p.$$
By using (5.1) and proceeding in a way similar to the preceding proof, we obtain
\[ \int_B |f|^{p\delta^r} \, dV \lesssim \int_B |\nabla f|^{p\delta^{r+1}} \, dV + |f(0)|^p. \]

Let \( j \geq 1 \) be an integer. Then, replacing \( f \) by \( \partial^\alpha f \), \( |\alpha| = j \), and taking \( r = jq \) in the above, we have
\[ \sum_{|\alpha|=0}^j \int_B |\partial^\alpha f|^p \, dV \lesssim \sum_{|\alpha|=0}^j \int_B |\partial^\alpha f|^p \, dV + \sum_{|\alpha|=0}^j |\partial^\alpha f(0)|^p \]
and therefore \( \|f\|_{p,j,3} \leq \|f\|_{p,j+1,3} \). Consequently, \( \|f\|_{p,m,1} \leq \sum_{j=1}^m \|f\|_{p,j,3} \leq \|f\|_{p,m,2} \) as desired. \( \square \)

**Proof of \( \|f\|_p \approx \|f\|_{p,m,2} \):** So far, we’ve seen that \( \|f\|_p \approx \|f\|_{p,j,1} \approx \|f\|_{p,j,3} \) for each \( j \geq 1 \). Thus, it is clear that \( \|f\|_{p,m,2} \lesssim \sum_{j=1}^m \|f\|_{p,j,3} \approx \|f\|_p \). For the other direction, let \( T_{ij} \) be any tangential differential operator and \( |\alpha| = 2m - 1 \). Then, by (5.5) with \( r = p(2m - 1) \), we have
\[ \int_B |\partial^{2m} T_{ij} f|^p \, dV \lesssim \int_B |\partial^{2m} T_{ij} f|^p \delta^{p(2m-1)} \, dV \]
\[ \lesssim \int_B |T_{ij} f|^p \delta^{p(2m-1)} \, dV. \]

Accordingly, we have
\[ \|f\|_{p,2m,2} \lesssim \|f\|_{p,2m-2}. \]

Let \( E_m \) be the differential operator as in Lemma 5.1. Then, we have
\[ \|f\|_{p,2m,1} \lesssim \|f\|_{p,2m,2} + \|\partial^{2m} E_m f\|_p \]
by Lemma 5.1. Let \( 0 < a < 1 \). Then, by (5.5), we have
\[ \int_{|x| > a} |\partial^{2m} E_m f(x)|^p \, dx \lesssim a^p \int_B |\partial^{2m-1} E_m f|^p \, dV \lesssim a^p \int_B |f|^p \, dV. \]

Also, by Lemma 2.2 and Proposition 5.2, we have
\[ \sup_{|x| \leq 1-a} |\partial^{2m-1} E_m f(x)| \leq C \sup_{|x| \leq 1-a} |f(x)| \leq C \|f\|_{p,2m,2} \]
for some constants \( C = C(a) \). Therefore, we have
\[ \|\partial^{2m} E_m f\|_p \leq C_1 \|f\|_{p,2m,2} + C_2 a \|f\|_p \]
where \( C_1 = C_1(a) \) and \( C_2 \) is independent of \( a \). Therefore, since we already have \( \|f\|_p \approx \|f\|_{p,2m,1} \), we have by (5.7) and (5.8)
\[ \|f\|_p \leq C_3 \|f\|_{p,2m,2} + C_4 a \|f\|_p \]
where \( C_3 = C_3(a) \) and \( C_4 \) is independent of \( a \). Hence, taking \( a \) to be sufficiently small, we conclude from (5.6) and (5.9) that \( \|f\|_p \lesssim \|f\|_{p,m,2} \). \( \square \)
We now turn to the norm equivalence for the harmonic Bloch spaces. For positive integers \( m \), put
\[
\|f\|_{B,m,1} = |f(0)| + \|\delta^m D^m f\|_{\infty}
\]
\[
\|f\|_{B,m,2} = |f(0)| + \sum_{|\alpha| = m} |\delta^m \partial^\alpha f|_{\infty}
\]
\[
\|f\|_{B,m,3} = \sum_{|\alpha| < m} |\delta^\alpha f(0)| + \sum_{|\alpha| = m} |\delta^m \partial^\alpha f|_{\infty}
\]
for functions \( f \) harmonic on \( B \). These norms also turn out to be equivalent.

**Theorem 5.5.** Let \( m \) be a positive integer. Then, there are positive constants \( C_1, C_2, C_3, C_4 \) such that
\[
\|f\|_{B} \leq C_1 \|f\|_{B,m,1} \leq C_2 \|f\|_{B,m,2} \leq C_3 \|f\|_{B,m,3} \leq C_4 \|f\|_{B}
\]
for functions \( f \) harmonic on \( B \).

In the proof below \( f \) is a given harmonic function on \( B \).

**Proof of** \( \|f\|_{B} \approx \|f\|_{B,m,1} \): The inequality \( \|f\|_{B,m,1} \lesssim \|f\|_{B} \) is implicit in the proof of Theorem 3.3. We now prove \( \|f\|_{B} \lesssim \|f\|_{B,m,1} \). First, note by Theorem 3.3
\[
\|f\|_{B} = \|QT_m f\|_{B} \lesssim \|T_m f\|_{\infty} \lesssim \|\delta f\|_{\infty} + \sum_{j=1}^{m} \|\delta^j D^j f\|_{\infty}.
\]

Hence, it is sufficient to show that the rightmost side of the above is dominated by \( \|f\|_{B,m,1} \). Let \( x \in B \) and \( |x| \geq 1/2 \). We write \( x = |x| \zeta \) where \( \zeta \in \partial B \). First, note (5.10)
\[
|f(\zeta/2)| \lesssim |f(0)| + \sup_{|y| \leq 3/4} |Df(y)|
\]
by Proposition 5.2. Let \( j \geq 1 \) be an integer. Then, since
\[
|f(x) - f(\zeta/2)| \leq 2 \int_{1/2}^{1} |Df(t\zeta)| \, dt,
\]
we have by (5.10)
\[
|f(x)| \lesssim |f(0)| + \|\delta^{j+1} Df\|_{\infty} \left( 1 + \int_{0}^{1} \frac{dt}{(1-t)^{j+1}} \right).
\]
This yields
\[
\sup_{|x| \geq 1/2} \delta^j |f(x)| \lesssim |f(0)| + \|\delta^{j+1} Df\|_{\infty},
\]
which, in turn, yields
\[
\|\delta^{j} f\|_{\infty} \lesssim |f(0)| + \|\delta^{j+1} Df\|_{\infty}.
\]
Taking \( j = 1 \), we obtain \( \|\delta f\|_{\infty} \lesssim |f(0)| + \|\delta Df\|_{\infty} \). Also, for \( 1 \leq j < m \), applying the above to \( D^j f \) and iterating, we obtain \( \|\delta^j D^j f\|_{\infty} \lesssim \|\delta^m D^m f\|_{\infty} \) because \( D^j f(0) = 0 \). We therefore conclude
\[
\|\delta f\|_{\infty} + \sum_{j=1}^{m} \|\delta^j D^j f\|_{\infty} \lesssim \|f\|_{B,m,1}
\]
which completes the proof. \( \Box \)
Theorem 5.6. Let \(|\alpha| = j \geq 1\), we have

\[
|\partial^\alpha f(x)| \leq |\partial^\alpha f(0)| + \int_0^{[x]} |\nabla \partial^\alpha f(tx/|x|)| \, dt
\]

and

\[
\int_0^{[x]} |\nabla \partial^\alpha f(tx/|x|)| \, dt \lesssim \|f\|_{B, j+1, 3} \int_0^{[x]} \frac{dt}{(1-t)^{j+1}}
\]

\[
\lesssim \|f\|_{B, j+1, 3} \delta^{-j}(x)
\]

for all \(x \in B\). This yields \(\|f\|_{B, j, 3} \lesssim \|f\|_{B, j+1, 3}\) for each \(j \geq 1\). Hence,

\[
\|f\|_B \approx \|f\|_{B, 1, 3} \lesssim \|f\|_{B, m, 3}.
\]

as desired.

Proof of \(\|f\|_B \approx \|f\|_{B, m, 2}\): We already have \(\|f\|_B \approx \|f\|_{B, j, 1} \approx \|f\|_{B, j, 3}\) for each \(j \geq 1\). Thus, it is clear that

\[
\|f\|_{B, m, 2} \lesssim \sum_{j=1}^m \|f\|_{B, j, 3} \approx \|f\|_B.
\]

Imitating the proof of \(\|f\|_B \approx \|f\|_{B, m, 2}\) of Theorem 5.4, we also have

\[
\|f\|_{B, 2m, 2} \lesssim \|f\|_{B, 2m-1, 2}
\]

and

\[
\|f\|_B \approx \|f\|_{B, 2m, 1} \leq C_1 \|f\|_{B, 2m, 2} + C_2 \|f\|_B
\]

where \(C_1 = C_1(a)\) and \(C_2\) is independent of \(a\). Hence, taking \(a\) to be sufficiently small, we obtain \(\|f\|_B \lesssim \|f\|_{B, m, 2}\). \(\square\)

We now close the paper with the corresponding little-o version.

Theorem 5.6. Let \(m\) be a positive integer and \(f \in B\). Then, the following conditions are all equivalent.

1. \(f \in B_0\).
2. \(\delta^m \partial^\alpha f \in C_0\).
3. \(\delta^m \partial^\alpha f \in C_0\) for all \(\alpha\) with \(|\alpha| = m\).
4. \(\delta^m \partial^\alpha f \in C_0\) for all \(\alpha\) with \(|\alpha| = m\).

Proof. The implication (1) \(\Rightarrow\) (4) is implicit in the proof of Theorem 3.3. We show (4) \(\Rightarrow\) (2). Assume \(\delta^m \partial^\alpha f \in C_0\) for \(|\alpha| = m\). Consider \(\alpha\) such that \(|\alpha| = m - 1\). Then, by (5.11),

\[
\limsup_{|\alpha| \to 1} \frac{\delta^m - 1}{(x) \partial^\alpha f(x)} \leq \limsup_{|\alpha| \to 1} \frac{\delta^m - 1}{(x)} \int_0^{[x]} |\nabla \partial^\alpha f(tx/|x|)| \, dt
\]

\[
= \limsup_{|\alpha| \to 1} \frac{\delta^m - 1}{(x)} \int_0^{[x]} |\nabla \partial^\alpha f(tx/|x|)| \, dt
\]

\[
\lesssim \sup_{|x| \geq r} \delta^m (y) |\nabla \partial^\alpha f(y)|
\]
for every positive \( r < 1 \). Hence, taking the limit \( r \to 1 \), we have \( \delta^{m-1}\partial^m f \in C_0 \).

Repeating the same, we obtain \( \delta^{m-1}\partial^m f \in C_0 \) for \( |\alpha| = m \) and thus \( \delta^m\partial^m f \in C_0 \).

We show (2) \( \implies \) (1). Assume \( \delta^m\partial^m f \in C_0 \). We first show that \( \delta^j \partial^j f \in C_0 \) for each \( j = 1, \ldots, m - 1 \). We only need consider the case \( j = m - 1 \). So, assume \( m \geq 2 \). Let \( x \in B \) and write \( x = |x|\zeta \) where \( \zeta \in \partial B \). Assume \( |x| > 1/2 \) and choose any \( r \) such that \( 1/2 < r < |x| \). We have

\[
|D^{m-1}f(x) - D^{m-1}f(r\zeta)| \leq r^{-1} \int_r^{|x|} |D^m f(t\zeta)| \, dt
\]

\[
\leq \left( \sup_{|t| \geq r} |\delta^m(y)D^m f(y)| \right) \int_r^{|x|} \frac{dt}{(1-t)^m}
\]

\[
\leq \delta^{m-1}(x) \left( \sup_{|t| \geq r} |\delta^m(y)D^m f(y)| \right)
\]

so that

\[
\delta^{m-1}(x)|D^{m-1}f(x)| \leq \delta^{m-1}(x)|D^{m-1}f(r\zeta)| + \sup_{|t| \geq r} |\delta^m(y)D^m f(y)|.
\]

Take the limit \( |x| \to 1 \) with \( r \) fixed and get

\[
\limsup_{|x| \to 1} \delta^{m-1}(x)|D^{m-1}f(x)| \leq \sup_{|t| \geq r} |\delta^m(y)D^m f(y)|.
\]

Since \( r > 1/2 \) is arbitrary and \( \delta^m\partial^m f \in C_0 \) by assumption, the above yields \( \delta^m\partial^m f \in C_0 \), as desired.

Now, since

\[
|T_m f| \leq \delta \sum_{j=0}^{m-1} |\delta^j \partial^j f| + \sum_{j=1}^m |\delta^j \partial^j f|
\]

\[
\leq \delta |f| + \|f\|_B + \sum_{j=1}^m |\delta^j \partial^j f|
\]

\[
\leq \delta (1 + \log \delta^{-1})\|f\|_B + \sum_{j=1}^m |\delta^j \partial^j f|
\]

we have \( T_m f \in C_0 \). We conclude \( f = QT_m f \in B_0 \) by Theorem 3.3, because \( Q \) maps \( C_0 \) onto \( B_0 \).

Since we already have (1) \( \iff \) (4), the implication (1) \( \implies \) (3) is clear. We now prove (3) \( \implies \) (1). Suppose (3) holds. By (5.4) (with \( p = |\alpha| = 1, r = 2m - 1 \), we have

\[
\delta^{2m}(x)|T_\alpha T^\alpha f(x)| \leq \sup_{y \in B_0} \delta^{2m-1}(y)|T^\alpha f(y)|
\]

for all \( x \in B \), \( T_\alpha \) and \( \alpha \) with \( |\alpha| = 2m - 1 \). This yields

\[
\limsup_{|\alpha| \to 1} \delta^{2m}(x) \sum_{|\alpha| = 2m} |T^\alpha f(x)| \leq \limsup_{|\alpha| \to 1} \delta^{2m-1}(x) \sum_{|\alpha| = 2m} |T^\alpha f(x)|
\]

and thus, without loss of generality, we may assume \( \delta^{2m} \sum_{|\alpha| = 2m} |T^\alpha f| \in C_0 \). Let \( E_m \) be the differential operator as in Lemma 5.1. Since \( \|\delta^{2m} E_m f\| \leq \|f\|_B \), we have \( \delta^{2m} E_m f \in C_0 \) and therefore \( \delta^{2m} \partial^{2m} f \in C_0 \) by Lemma 5.1. Since (1) \( \iff \) (2), it follows that \( f \in B_0 \).
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