COMMUTING TOEPLITZ OPERATORS WITH PLURIHARMONIC SYMBOLS ON THE FOCK SPACE

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ABSTRACT. In the setting of the Bergman space over the disk or the ball, it has been known that two Toeplitz operators with bounded pluriharmonic symbols can (semi-)commute only in the trivial cases. In this paper we study the analogues on the Fock space over the multi-dimensional complex space. As is the case in various other settings, we are naturally led to the problem of characterizing certain type of fixed points of the Berezin transform. For such fixed points, we obtain a complete characterization by means of eigenfunctions of the Laplacian. We also obtain other characterizations. In particular, it turns out that there are many nontrivial cases on the Fock space for (semi-)commuting Toeplitz operators with pluriharmonic symbols. All in all our results reveal that the situation on the Fock space appears to be much more complicated than that on the classical Bergman space setting, which partly is caused by the unboundedness of the operator symbols. Some of our results are restricted to the one-variable case and the corresponding several-variable case is left open.

1. Introduction

In 1964 Brown and Halmos [9] showed that Toeplitz operators with bounded symbols, acting on the Hardy space over the unit disk, can commute only in the trivial cases. Initiated by such a seminal work of Brown and Halmos, the problem of characterizing when two Toeplitz operators commute has been one of the topics of constant interest in the study of Toeplitz operators on classical function spaces over various domains. The problem with arbitrary bounded symbols being still far from its solution, all the results known so far (except for [9]) are restricted to certain subclasses of symbols; see, for example, [5, 6, 7, 10, 12, 13, 17, 19, 20, 24, 27] and references therein. In particular, for Toeplitz operators with bounded harmonic symbols over the unit disk, Axler and Čučković [2] obtained the complete Bergman space analogue of the aforementioned result of Brown and Halmos. Later Zheng [27] extended these results to Toeplitz operators with bounded pluriharmonic symbols acting on the Hardy/Bergman space over the ball. The purpose of the current paper is to explore commuting Toeplitz operators with pluriharmonic symbols acting on the Fock space described below.

We first recall the function space to work on. Let \( n \) be a fixed positive integer, reserved for the dimension of the underlying complex \( n \)-space \( \mathbb{C}^n \). For \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \) in \( \mathbb{C}^n \), we write \( z \cdot w = \sum_{j=1}^{n} z_j w_j \) so that \( z \cdot \overline{w} \) is the Hermitian inner product.
of \( z, w \in \mathbb{C}^n \) and \( |z| = (z \cdot z)^{1/2} \). We denote by \( d\mu \) the normalized Gaussian measure on \( \mathbb{C}^n \) given by
\[
d\mu(z) := \frac{1}{\pi^n} e^{-|z|^2} dv(z)
\]
where \( dv = dv_n \) denotes the Lebesgue measure on \( \mathbb{C}^n \). The Fock space \( H^2(\mathbb{C}^n) \) is then the space of all Gaussian square integrable entire functions on \( \mathbb{C}^n \), i.e.,
\[
H^2(\mathbb{C}^n) := L^2(\mathbb{C}^n, d\mu) \cap Hol(\mathbb{C}^n).
\]
Here, and elsewhere, \( Hol(\cdot) \) stands for the class of functions holomorphic over the domain specified in the parenthesis. As is well known, \( H^2(\mathbb{C}^n) \) is a closed subspace of \( L^2(\mathbb{C}^n, d\mu) \) and thus is a Hilbert space. In the literature it frequently is denoted as Bargmann space or Segal-Bargmann space. The above naming is due to the fact that \( H^2(\mathbb{C}^n) \) canonically is isomorphic to the usual Fock space over \( \mathbb{C}^n \). We refer the reader to [29] for a systematic treatment of both function theoretic and operator theoretic aspects of the one-dimensional Fock spaces.

We now recall the operators to work with. Given a complex valued measurable function \( u \) on \( \mathbb{C}^n \) satisfying suitable growth condition at \( \infty \), the Toeplitz operator \( T_u \) with symbol \( u \) is defined by
\[
T_u := PM_u
\]
where \( P : L^2(\mathbb{C}^n, d\mu) \to H^2(\mathbb{C}^n) \) denotes the Hilbert space orthogonal projection and \( M_u \) denotes the operator of pointwise multiplication by \( u \). In general the symbol function of the Toeplitz operator in (1.1) is unbounded and therefore may lead to an unbounded operator on \( H^2(\mathbb{C}^n) \). We will specify later (see Section 2.2) the class of symbols, denoted by Sym(\( \mathbb{C}^n \)), which is not small in the sense that it contains all the functions with at most linear exponential growth at \( \infty \). Our symbol class is an algebra and has the property that products of finitely many Toeplitz operators with symbols therein are always densely defined on \( H^2(\mathbb{C}^n) \). In particular, given two Toeplitz operators with symbols in Sym(\( \mathbb{C}^n \)), their (semi-)commutators are densely defined on \( H^2(\mathbb{C}^n) \).

Recall that a complex valued function on \( \mathbb{C}^n \) is said to be pluriharmonic if its restriction to an arbitrary complex line is harmonic as a function of one complex variable. As is well known, a pluriharmonic function on \( \mathbb{C}^n \) is precisely a function which can be expressed as sum of a holomorphic function and a co-holomorphic function on \( \mathbb{C}^n \); see, for example, [18, Proposition 2.2.3]. Our main aim in this paper is to explore the problem of when two Toeplitz operators with pluriharmonic symbols (semi-)commute. The start line for our approach is the same as the one for earlier studies on the Bergman space over the classical bounded domains such as the disk ([2]), the ball ([27]) and the polydisk ([10]). That is, we convert the original operator theoretic problem to a function theoretic one via the Berezin transform \( B \); see Section 2.3 for precise definition. In fact, the Berezin transform converts our problem to that of characterizing \( \mathcal{B} \)-fixed points of the form \( f\overline{g} + h\overline{k} \) with entire functions \( f, g, h, k \in \text{Sym}(\mathbb{C}^n) \); see Section 2.5. However, we immediately encounter the main difficulty for our Fock space setting: \( \mathcal{B} \)-fixed points might not have any harmonicity in general, unlike the aforementioned Bergman space setting.

In the next section we collect some known material and prove some basic facts that will be needed subsequently. In Section 3, in connection with the remarks in the preceding
paragraph, we study $\mathcal{B}$-fixed points of the form

\begin{equation}
\sum_{j=1}^{N} f_j \overline{g_j}
\end{equation}

where all functions $f_j, g_j \in \text{Sym}(\mathbb{C}^n)$ are entire. We first handle the simplest case where $N = 1$ and functions involved are both polynomials; see Theorem 3.2. We then characterize functions in the algebra generated by the holomorphic polynomials and the reproducing kernels; see Theorem 3.5. This result reveals an intimate relation between eigenfunctions of the Laplacian and the $\mathcal{B}$-fixed points of the form under consideration. We are thus eventually led to the “eigenfunction-series” characterization of $\mathcal{B}$-fixed points of the form (1.2); see Theorem 3.11. In the course of the proofs we obtain a characterization (Corollary 3.12) by harmonicity when each pair $\{f_j, g_j\}$ contains a polynomial.

In Section 4, we apply our results obtained in earlier sections to study the one-variable case of (semi-)commuting Toeplitz operators with harmonic functions whose (co-)holomorphic parts have at most linear exponential growth at $\infty$. This amounts to studying $\mathcal{B}$-fixed points of the form (1.2) with $N \leq 2$ where functions involved all have the specified growth rate at $\infty$. We obtain quite explicit characterizations other than the eigenfunction-series characterization mentioned above; see Theorems 4.1 and 4.14. These characterizations show that there are extra cases for the Fock space, which have no analogue on the Bergman space over the ball or the polydisk. The several-variable case appears to be more subtle and is left open. As an application of our results, we obtain a “zero-product property” (Theorem 4.2) for Toeplitz operators under consideration. In addition, we provide a counter example (Example 4.3) to the analogue of a question posed by Louhichi and Rao.

2. Preliminaries

In this section we recall some known results and prove some basic facts related to the symbol class and the Berezin transform.

2.1. Auxiliary Fock spaces. Consider a family of normalized Gaussian measures on $\mathbb{C}^n$ given by

\[ d\mu_t(z) := \frac{1}{(t\pi)^n} e^{-\frac{|z|^2}{4t}} dv(z) \]

for $t > 0$. Associated with the measure $d\mu_t$ is the $t$-scaled Fock space

\[ H^2_t(\mathbb{C}^n) := L^2(\mathbb{C}^n, d\mu_t) \cap \text{Hol}(\mathbb{C}^n), \]

which is regarded as a closed subspace of $L^2(\mathbb{C}^n, d\mu_t)$. We write $\langle \cdot, \cdot \rangle_t$ and $\| \cdot \|_t$ for the inner product and norm of $L^2(\mathbb{C}^n, d\mu_t)$, respectively. As is well known, $H^2_t(\mathbb{C}^n)$ is a reproducing kernel Hilbert space whose reproducing kernel $K^t_w$ at $w \in \mathbb{C}^n$ is given by

\[ K^t_w(z) = e^{\frac{z \cdot \overline{w}}{t}}; \]

see, for example, [3, 29]. Note $\| K^t_w \|_t^2 = K^t_w(w) = e^{\frac{|w|^2}{t}}$ by the reproducing property. We denote by

\[ k^t_w(z) := \frac{K^t_w(z)}{\| K^t_w \|_t} = \exp \left\{ \frac{z \cdot \overline{w}}{t} - \frac{|w|^2}{2t} \right\} \]
the normalized kernel at \( w \). If we omit the index \( t \) in these notation, we always mean \( t = 1 \). We note a simple but useful integral identity

\[
\int_{\mathbb{C}^n} e^{2\text{Re}(z \cdot w)} d\mu_t(w) = e^{t|z|^2},
\]

which can be easily verified via the relation \( K_w = K_t^t \).

2.2. Symbol space. Given \( c > 0 \), consider the space \( D_c \) of all complex measurable functions \( u \) on \( \mathbb{C}^n \) such that \( u(z)e^{-c|z|^2} \) is essentially bounded on \( \mathbb{C}^n \). Clearly, each \( D_c \) is a Banach space equipped with the norm \( \|u\|_{D_c} := \|ue^{-c|\cdot|^2}\|_{L^\infty(\mathbb{C}^n,dv)} \). Using these spaces, we now define our symbol space by

\[
\text{Sym}(\mathbb{C}^n) := \bigcap_{c > 0} D_c.
\]

Note that \( \text{Sym}(\mathbb{C}^n) \) is a \(*\)-algebra under pointwise multiplication and complex conjugation. Given \( 0 < s < t \), note that there is a constant \( C_{s,t} > 0 \) such that

\[
C_{s,t}\|f\|_s \leq \|f\|_{D_{s/2t}} \leq \|f\|_t
\]

for \( f \in H^2_t(\mathbb{C}^n) \); the first inequality is clear and the second one can be easily verified via the reproducing property. It follows that

\[
\text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n) = \bigcap_{t > 0} H^2_t(\mathbb{C}^n).
\]

We equip \( \text{Sym}(\mathbb{C}^n) \) with the Fréchet topology which is induced by the system of norms \( \{p_j := \|\cdot\|_{D_{1/2^j}}\}_{j \in \mathbb{N}} \); the symbol \( \mathbb{N} \) denotes the set of all positive integers. While we refer to [23] for the precise notion of Fréchet topology and related facts, we would like to mention here that \( \text{Sym}(\mathbb{C}^n) \) now has the structure of a complete metric space with the property: a sequence in \( \text{Sym}(\mathbb{C}^n) \) converges if and only if it converges with respect to each of the above norms \( p_j \), \( j \in \mathbb{N} \).

Consider the vector space of entire function

\[
\mathcal{H}(\mathbb{C}^n) := \left( \bigcup_{0 < c < \frac{1}{2}} D_c \right) \cap \text{Hol}(\mathbb{C}^n).
\]

Clearly, \( \mathcal{H}(\mathbb{C}^n) \) is contained in \( H^2(\mathbb{C}^n) \) and contains all the reproducing kernels of \( H^2(\mathbb{C}^n) \). We thus see that \( \mathcal{H}(\mathbb{C}^n) \) is a dense subset of \( H^2(\mathbb{C}^n) \). It has been shown in [4] that \( \mathcal{H}(\mathbb{C}^n) \) is invariant under the action of each Toeplitz operator with symbol in \( \text{Sym}(\mathbb{C}^n) \). So, \( \mathcal{H}(\mathbb{C}^n) \) is invariant under products of finitely many such operators. In particular, products of finitely many Toeplitz operators with symbols in \( \text{Sym}(\mathbb{C}^n) \) are always densely defined on \( H^2(\mathbb{C}^n) \).

In what follows we use the standard multi-index notation. That is, for \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \) where \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), we use the notation

\[
|\alpha| := \alpha_1 + \cdots + \alpha_n, \quad \alpha! := \alpha_1! \cdots \alpha_n!, \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]

for \( x = (x_1, \ldots, x_n) \). Setting \( \partial := (\partial_1, \ldots, \partial_n) \)
where $\partial_j$ denotes the differentiation with respect to the $j$-th component of given variable, we also use the notation

$$ f^{(\alpha)} := \partial^\alpha f $$

for $f \in Hol(\mathbb{C}^n)$. Finally, we will use the scaled Laplacian

$$ \Delta := \partial \cdot \overline{\partial} = \sum_{j=1}^n \partial_j \overline{\partial}_j $$

which differs from the standard one by a factor of 4.

**Lemma 2.1.** If $f \in \text{Sym}(\mathbb{C}^n) \cap Hol(\mathbb{C}^n)$, then $f^{(\alpha)} \in \text{Sym}(\mathbb{C}^n)$ for all $\alpha \in \mathbb{N}_0^n$.

**Proof.** Let $f \in \text{Sym}(\mathbb{C}^n) \cap Hol(\mathbb{C}^n)$ and $\alpha \in \mathbb{N}_0^n$. Let $\varepsilon > 0$. Consider $t > (2\varepsilon)^{-1}$ to begin with. Given $z \in \mathbb{C}^n$, we have by the reproducing property

$$ f(z) = \langle f, K_t^* \rangle_t = \int_{\mathbb{C}^n} f(w) e^{\frac{z \cdot w}{t}} d\mu_t(w) $$

and thus differentiation under the integral sign yields

$$ f^{(\alpha)}(z) = \frac{1}{t^{l(\alpha)}} \int_{\mathbb{C}^n} \overline{w}^\alpha f(w) e^{\frac{z \cdot w}{t}} d\mu_t(w). $$

Note that $|f(z)| \leq \|f\|_{\mathcal{D}_\varepsilon} e^{\varepsilon|z|^2}$. Also, note that there is a constant $C = C(|\alpha|, \varepsilon) > 0$ and

$$ |\overline{w}^\alpha| \leq |w|^{|\alpha|} \leq C e^{\varepsilon|w|^2} $$

for all $w \in \mathbb{C}^n$. It follows from these observations and (2.4) that

$$ |f^{(\alpha)}(z)| \leq \frac{C \|f\|_{\mathcal{D}_\varepsilon}}{\pi^{n|\alpha|+n}} \int_{\mathbb{C}^n} \exp \left\{ \frac{\text{Re} (z \cdot w)}{t} - \left( \frac{1}{t} - 2\varepsilon \right) |w|^2 \right\} dv(w) $$

$$ = \frac{C \|f\|_{\mathcal{D}_\varepsilon}}{t^{l(\alpha)}(1-2\varepsilon)^n} \exp \left\{ \frac{|z|^2}{4t(1-2\varepsilon t)} \right\}; $$

the last equality comes from (2.1). In particular, choosing $t := \frac{1}{4\varepsilon}$, we have

$$ |f^{(\alpha)}(z)| \leq 2^n (4\varepsilon)^{|\alpha|} C \|f\|_{\mathcal{D}_\varepsilon} e^{2\varepsilon|z|^2} $$

so that

$$ \|f^{(\alpha)}\|_{\mathcal{D}_\varepsilon} \leq 2^n (4\varepsilon)^{|\alpha|} C \|f\|_{\mathcal{D}_\varepsilon}. $$

Since $\varepsilon > 0$ is arbitrary, this yields $f^{(\alpha)} \in \text{Sym}(\mathbb{C}^n)$, as required. \hfill $\square$

**2.3. Heat transform.** Given $t > 0$ and $u \in \text{Sym}(\mathbb{C}^n)$, define

$$ B_t[u](z) := \int_{\mathbb{C}^n} u(z \pm w) d\mu_t(w) $$

$$ = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} u(w) \exp \left\{ -\frac{|z - w|^2}{t} \right\} dv(w) $$

for $z \in \mathbb{C}^n$. Put

$$ \Lambda_u(t, z) := B_t[u](z) $$

for $t > 0$ and $z \in \mathbb{C}^n$. 
As is well known, the function $\Lambda_u$ satisfies the heat equation
\begin{equation}
(\Delta_z - \partial_t) \Lambda_u(t, z) = 0
\end{equation}
where $\partial_t = \frac{\partial}{\partial t}$ and $\Delta_z$ denotes the Laplacian with respect to $z$-variable; see [8] or, for the one-variable case, [29, Theorem 3.14]. For this reason $B_t[u]$ is often called the heat transform of $u$ (at time $t$). Using the mean value property one may easily check that harmonic functions in the symbol space are fixed by the heat transform. Also is well known that the Laplacian commutes with the heat transform when applied to functions which together with their derivatives have a suitable growth rate at $\infty$. For example, that is the case when they act on the second-order Sobolev type subspace
\begin{equation}
\text{Sym}_2(C^n) := \left\{ u \in C^2(C^n) : u, \partial_j u, \partial_j \overline{\partial_j} u \in \text{Sym}(C^n) \text{ for all } j = 1, \ldots, n \right\}
\end{equation}
of the symbol space. In fact, for $u \in \text{Sym}_2(C^n)$, using the identity
\begin{equation}
\Delta_z \left[ \exp \left\{ -|z - w|^2/t \right\} \right] = \Delta_w \left[ \exp \left\{ -|z - w|^2/t \right\} \right],
\end{equation}
one may check via two times partial integration in (2.5) (the boundary terms vanish due to the growth assumption on the derivatives of $u$) that
\begin{equation}
\Delta B_t[u] = B_t[\Delta u]
\end{equation}
for $t > 0$. The next lemma shows that the symbol space is invariant under the heat transform.

**Lemma 2.2.** Given $\varepsilon, t > 0$ with $0 < \varepsilon \leq \frac{1}{2t}$, the inequality
\begin{equation}
\|B_t[u]\|_{D_{2\varepsilon}} \leq 2^n \|u\|_{D_{\varepsilon}}
\end{equation}
holds for $u \in \text{Sym}(C^n)$. In particular, for all $\alpha, \beta \in \mathbb{N}_0^n$ and $t > 0$, the operators
\begin{equation}
\partial^\alpha \overline{\partial^\beta} B_t : \text{Sym}(C^n) \to \text{Sym}(C^n)
\end{equation}
are well-defined and continuous in the Fréchet topology of $\text{Sym}(C^n)$.

**Proof.** Let $0 < \varepsilon \leq \frac{1}{2t}, u \in \text{Sym}(C^n)$ and fix $z \in C^n$. Note $|u(z)| \leq \|u\|_{D_{\varepsilon}} e^{\varepsilon |z|^2}$. Thus (2.5) yields
\begin{equation}
|B_t[u](z)| \leq \frac{\|u\|_{D_{\varepsilon}}}{(\pi t)^n} \int_{C^n} \exp \left\{ \varepsilon |z + \xi|^2 - \frac{|\xi|^2}{t} \right\} dv(\xi)
\end{equation}
\begin{equation}
= \frac{\|u\|_{D_{\varepsilon}} e^{\varepsilon |z|^2}}{(\pi t)^n} \int_{C^n} \exp \left\{ 2\varepsilon \text{Re}(z \cdot \overline{\xi}) - \left( \frac{1}{t} - \varepsilon \right) |\xi|^2 \right\} dv(\xi)
\end{equation}
\begin{equation}
= \frac{\|u\|_{D_{\varepsilon}}}{(1 - t\varepsilon)^n} \exp \left\{ \varepsilon \left( 1 + \frac{t\varepsilon}{1 - t\varepsilon} \right) |z|^2 \right\};
\end{equation}
we used (2.1) for the last equality. Since $0 < t\varepsilon \leq 1/2$, this implies the asserted inequality.

In the case where $\alpha = \beta = 0$ the second part of the lemma is immediate from the first part. The statement for general $\alpha, \beta \in \mathbb{N}_0^n$ follows from this special case, differentiation under the integral sign and the fact that multiplication by polynomials in $z$ and $\overline{z}$ is a continuous operation on $\text{Sym}(C^n)$.

**Remark 2.3.** With our previous notation, it follows from Lemma 2.2 that each $B_t$ maps the symbol space $\text{Sym}(C^n)$ into $\text{Sym}_2(C^n)$. 

\begin{proof}
\end{proof}
Now, it is legitimate by Lemma 2.2 to consider $B_t \circ B_s$ for any $s, t > 0$ on the symbol space. In conjunction with this remark, we note the well-known semi-group property

$$(2.9) \quad B_t \circ B_s = B_{t+s};$$

see, for example, [8] or, for the one-variable case, [29, Theorem 3.13]. When $t = 1$, we write $B := B_1$. Note

$$|k_z(w)|^2 = \exp \left\{ 2\Re (z \cdot \overline{w}) - |z|^2 \right\} = e^{|w|^2 - |z-w|^2}$$

and thus

$$B[u](z) = \left\langle uk_z, k_z \right\rangle = \left\langle P(uk_z), k_z \right\rangle = \left\langle T_uk_z, k_z \right\rangle$$

for $u \in \text{Sym}(\mathbb{C}^n)$. This relation allows us to extend the definition of $B$ to operators as follows. Let $\mathcal{T}$ be the algebra consisting of all finite sums of finite products of Toeplitz operators with symbols in $\text{Sym}(\mathbb{C}^n)$. Given $T \in \mathcal{T}$ we define

$$B[T](z) := \left\langle Tk_z, k_z \right\rangle$$

for $z \in \mathbb{C}^n$. We call $B[T]$ the Berezin transform of $T$. Injectivity of the Berezin transform is well known.

**Lemma 2.4.** The Berezin transform $B : \mathcal{T} \rightarrow \mathcal{C}^\omega(\mathbb{C}^n)$ is one-to-one. Here, $\mathcal{C}^\omega(\mathbb{C}^n)$ denotes the class of all real analytic functions on $\mathbb{C}^n$.

We refer to [4, Lemma 12] for a proof of the above lemma.

### 2.4. Periodic entire functions.

Given $u \in \text{Sym}(\mathbb{C}^n)$, consider the function $\Lambda_u$ on $(0, \infty) \times \mathbb{C}^n$ as in (2.6). From the growth condition of $u$ and (2.5) it is easy to see that the function $\Lambda_u(\cdot, z)$ with $z \in \mathbb{C}^n$ fixed holomorphically extends to the right half-plane $\{ \lambda \in \mathbb{C} : \Re \lambda > 0 \}$. Assume now that $u$ is fixed under the Berezin transform, i.e., $B[u] = u$. It then follows from the semigroup property (2.9) that

$$(2.10) \quad \Lambda_u(t + 1, z) = B_{t+1}[u](z) = (B_t \circ B)[u](z) = B_t[u](z) = \Lambda_u(t, z)$$

for all $t > 0$ and $z \in \mathbb{C}^n$. By the identity theorem we find that $\Lambda_u(\cdot, z)$ defines a 1-periodic function (i.e. a periodic function with period 1) on the right half-plane and therefore extends to a periodic entire function on the complex plane.

In conjunction with the observation in the preceding paragraph, we recall the characterization of periodic entire functions on $\mathbb{C}$ of at most linear exponential growth at $\infty$. We say that a measurable function $h$ on $\mathbb{C}^n$ is of exponential type and write $h \in \mathcal{E}(\mathbb{C}^n)$, if there is some $s > 0$ such that $|h(z)| e^{-s|z|}$ is essentially bounded on $\mathbb{C}^n$. Clearly, we have

$$\mathcal{E}(\mathbb{C}^n) \subset \text{Sym}(\mathbb{C}^n).$$

A short proof of the next (well-known) lemma can be found in [4].

**Lemma 2.5.** Let $h \in \text{Hol}(\mathbb{C}) \cap \mathcal{E}(\mathbb{C})$ be 1-periodic. Then $h$ is a trigonometric polynomial, i.e., of the type

$$h(\lambda) = \sum_{m=-N}^{N} c_m e^{2\pi im\lambda} \quad (i = \sqrt{-1})$$

for some $N \in \mathbb{N}_0$ and coefficients $c_m$.

If the “exponential type” hypothesis is relaxed in the hypothesis of Lemma 2.5, we get a trigonometric series rather than a trigonometric polynomial.
Proposition 2.6. Let \( h \in \text{Hol}(\mathbb{C}) \) be 1-periodic. Then \( h \) is a trigonometric series, i.e., of the type
\[
h(\lambda) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m \lambda}
\]
for some coefficients \( c_m \) which makes the series above converge absolutely and uniformly on each horizontal strip in \( \mathbb{C} \).

Proof. Define
\[
\tilde{h}(\lambda) := h\left(\frac{\text{Log} \lambda}{2\pi i}\right),
\]
where \( \text{Log} \lambda \) denotes the principal branch of the logarithmic function. Since \( h \) is 1-periodic, \( \tilde{h} \) extends to a holomorphic function on \( \mathbb{C} \setminus \{0\} \). For \( \lambda = x + iy \) with \( x \) and \( y \) real, note
\[
\tilde{h}(e^{2\pi i \lambda}) = \tilde{h}(e^{-2\pi y} e^{2\pi ix}) = h\left(\frac{-2\pi y + 2\pi ix}{2\pi i}\right) = h(\lambda);
\]
the second equality holds for \(-\frac{1}{2} < x < \frac{1}{2}\) by definition of \( \text{Log} \) and thus for general \( x \) by periodicity and continuity. Thus, using the Laurent expansion of \( \tilde{h} \) near the origin
\[
(2.11)\quad \tilde{h}(\lambda) = \sum_{m=-\infty}^{\infty} c_m \lambda^m, \quad \lambda \neq 0
\]
the original function \( h \) can be represented in the form
\[
(2.12)\quad h(\lambda) = \tilde{h}(e^{2\pi i \lambda}) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m \lambda}
\]
for \( \lambda \in \mathbb{C} \). The series above converges absolutely and uniformly on each horizontal strip, because the series in (2.11) converges absolutely and uniformly on each compact annular region in \( \mathbb{C} \setminus \{0\} \). This completes the proof. \( \square \)

2.5. (Semi-)commutators of Toeplitz operators. Given \( T_u \) and \( T_v \) with \( u, v \in \text{Sym}(\mathbb{C}^n) \), we denote by
\[
(T_u, T_v) := T_{uv} - T_u T_v
\]
and
\[
[T_u, T_v] := T_u T_v - T_v T_u
\]
the semi-commutator and the commutator of \( T_u \) and \( T_v \), respectively. Note that these operators are densely defined on \( H^2(\mathbb{C}^n) \).

In connection with Lemma 2.4, we compute the Berezin transform of (semi-)commutators of two Toeplitz operators with pluriharmonic symbols. First, let’s compute the Berezin transform of the (semi-)commutator \( (T_f, T_g) = [T_{\overline{g}}, T_f] \) for \( f, g \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n) \). For \( z \in \mathbb{C}^n \), we have by the reproducing property
\[
[T_{\overline{g}}K_z](\xi) = \langle gK_z, K_\xi \rangle = \langle gK_\xi, K_z \rangle = \overline{g(z)}K_z(\xi)
\]
for all \( \xi \in \mathbb{C}^n \). In other words, \( [T_{\overline{g}}K_z] = \overline{g(z)}K_z \). It follows that
\[
B[T_fT_{\overline{g}}](z) = \langle fT_{\overline{g}}k_z, k_z \rangle = \overline{g(z)}\langle fk_z, k_z \rangle = \overline{g(z)}f(z);
\]
the last equality holds, because the Berezin transform fixes holomorphic functions. Combining these observations, we obtain
\begin{equation}
B[T_f, T_{\bar{g}}] = B[f \bar{g}] - f \bar{g}.
\end{equation}
Next, consider a pair of symbol functions $u, v \in \text{Sym}(%C^n)$ of the form
\begin{equation}
u = f + k \quad \text{and} \quad v = h + \bar{g}
\end{equation}
where $f, g, h, k \in \text{Hol}(C^n)$. In fact one may check $f, g, h, k \in \text{Sym}(C^n)$ under the assumption $u, v \in \text{Sym}(C^n)$, because the growth rate of a holomorphic function is controlled by that of its real part.

A direct calculation yields $(T_v, T_u) = (T_h, T_k)$ and thus
\begin{equation}
[T_u, T_v] = (T_v, T_u) = (T_h, T_k) = (T_f, T_{\bar{g}}).
\end{equation}
We thus have by (2.13)
\begin{equation}
B[T_u, T_v] = B[h \bar{k} - f \bar{g}] + f \bar{g} - h \bar{k}.
\end{equation}
Now, the next proposition is immediate from (2.13), (2.15) and Lemma 2.4.

**Proposition 2.7.** Let $f, g, h, k \in \text{Sym}(C^n) \cap \text{Hol}(C^n)$. Then the following statements hold:
\begin{itemize}
\item[(a)] $(T_f, T_{\bar{g}}) = 0 \iff B[f \bar{g}] = f \bar{g}$.
\item[(b)] $[T_{f+k}, T_{h+\bar{g}}] = 0 \iff B[f \bar{g} - h \bar{k}] = f \bar{g} - h \bar{k}$.
\end{itemize}

**2.6. Miscellany.** By Proposition 2.7 we are naturally led to the study on $B$-fixed points. We note here a known result in this direction. Let $S(C^n)$ be the Schwartz space over $C^n \cong R^{2n}$. As is well known, the Berezin transform $B$ can be regarded as a continuous convolution operator
\begin{equation}
B : S(C^n) \longrightarrow S(C^n) : B[u] = u \ast h = (2\pi)^{-n} \int_{C^n} u(w) h(\cdot - w) \, dv(w),
\end{equation}
where $h = 2^n \exp\{-|\cdot|^2\}$. Therefore it naturally admits an extension to the dual space $S'(C^n)$ of tempered distributions. The $B$-fixed points in $S'(C^n)$ are completely characterized as in the next lemma; see [14].

**Lemma 2.8.** Let $u \in S'(C^n)$. If $B[u] = u$, then $u$ is a harmonic polynomial.

We also recall the following well-known “complexification” lemma; see, for example [16, Proposition 1.69].

**Lemma 2.9.** Let $\Omega$ be a domain in $C^n$ and assume that $Q$ is holomorphic on $\Omega \times \Omega^*$ where $\Omega^* = \{z : z \in \Omega\}$. If $Q(z, \bar{z}) = 0$ for all $z \in \Omega$, then $Q = 0$ on $\Omega \times \Omega^*$.

### 3. Fixed points of the Berezin transform

In this section, in view of Proposition 2.7, we explore solutions of the equation
\begin{equation}
B \left[ \sum_{\ell=1}^{N} f_{\ell} \bar{g}_{\ell} \right] = \sum_{\ell=1}^{N} f_{\ell} \bar{g}_{\ell}
\end{equation}
where $f_{\ell}, g_{\ell} \in \text{Sym}(C^n) \cap \text{Hol}(C^n)$ for each $\ell$. To begin with, we first consider the simplest case
\begin{equation}
B[f \bar{g}] = f \bar{g},
\end{equation}
where \( f, g \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n) \).

When \( f \) and \( g \) are polynomials, we see from Lemma 2.8 that (3.2) holds if and only if \( f \overline{g} \) is harmonic. Such type of harmonic functions are characterized as in the next lemma. In what follows, a function \( u \in \mathcal{O}(\mathbb{C}^n) \) is said to be \( n \)-harmonic if \( u \) is harmonic in each variable separately. Thus, \( f \overline{g} \) with \( f, g \in \text{Hol}(\mathbb{C}^n) \) is \( n \)-harmonic if and only if \( \partial_j f = 0 \) or \( \partial_j g = 0 \) for each \( j = 1, \ldots, n \).

**Lemma 3.1.** Given \( f, g \in \text{Hol}(\mathbb{C}^n) \), the following statements are equivalent:

(a) \( f \overline{g} \) is harmonic.
(b) \( (f \overline{g}) \circ U \) is \( n \)-harmonic for some unitary operator \( U \) on \( \mathbb{C}^n \).

**Proof.** Note that \( n \)-harmonicity implies harmonicity and that harmonicity is invariant under composition with unitary operators. Thus the implication \( (b) \implies (a) \) holds. Conversely, assume (a). Then we have \( \partial f(z) \cdot \overline{\partial g(z)} = 0 \) and thus by Lemma 2.9

\[
\partial f(z) \cdot \overline{\partial g(w)} = 0
\]

for all \( z, w \in \mathbb{C}^n \). So, setting

\[
S_f := \text{span} \{ \partial f(z) : z \in \mathbb{C}^n \} \quad \text{and} \quad S_g := \text{span} \{ \partial g(z) : z \in \mathbb{C}^n \},
\]

we see that \( S_f \perp S_g \). Now, put \( d := \dim S_f \), pick an orthonormal basis \( \{ B_1, \ldots, B_d \} \) of \( S_f \), and extend it to an orthonormal basis \( \{ B_1, \ldots, B_n \} \) of \( \mathbb{C}^n \). Let \( U \) be the \( n \times n \) matrix whose \( j \)-th column vector is \( B_j \). We then have

\[
\partial_j (f \circ U)(z) = ([\partial_j f](Uz)) \cdot \overline{B_j} = 0 \quad \text{for} \quad j > d,
\]

because \( \{ B_j \}_{j > d} \subset \mathbb{C}^n \ominus S_f \). Similarly, \( \partial_j (g \circ U) = 0 \) for \( j \leq d \). Accordingly, \( (f \overline{g}) \circ U \) is \( n \)-harmonic, as asserted. This completes the proof. \( \square \)

As an immediate consequences of Lemmas 2.8 and 3.1, we obtain the following characterization.

**Theorem 3.2.** Given polynomials \( p, q \in \text{Hol}(\mathbb{C}^n) \), the following statements are equivalent:

(a) \( B[p \overline{q}] = pq \).
(b) \( p \overline{q} \) is harmonic.
(c) \( (p \overline{q}) \circ U \) is \( n \)-harmonic for some unitary operator \( U \) on \( \mathbb{C}^n \).

When \( n = 1 \), the statements above reduce to

(d) either \( p \) or \( q \) is constant.

However, if the function class is broadened to a larger class, say the class of exponential type, the characterization by means of harmonicity is no longer true. To see an example on \( \mathbb{C} \), put

\[
h_{a,b}(z) = e^{a \text{Re} z + b \text{Im} z}
\]

for \( a, b \in \mathbb{C} \). One may compute

\[
B[h_{a,b}] = e^{\frac{a^2 + b^2}{4}} h_{a,b};
\]

see [14] or [29, pp. 114–115]. Also, note \( \Delta h_{a,b} = \frac{a^2 + b^2}{4} h_{a,b} \). Thus, if \( 0 \neq a^2 + b^2 \in 8\pi i \mathbb{Z} \), then \( h_{a,b} \) is non-harmonic but fixed by the Berezin transform. Here, and in what follows, the symbol \( \mathbb{Z} \) stands for the set of all integers.

We now proceed to the investigation into the solutions, suggested by the examples in the preceding paragraph, of (3.2). To be more explicit, we introduce some notation. Let \( \mathcal{P}_n \) be
the algebra of holomorphic polynomials on \( \mathbb{C}^n \) and let \( \mathcal{K}_n \) be the algebra generated by all reproducing kernels for \( H^2(\mathbb{C}^n) \). We denote by \( \mathcal{A}_n \) the algebra generated by \( \mathcal{P}_n \) and \( \mathcal{K}_n \). More explicitly, we have

\[
\mathcal{A}_n := \left\{ \sum_{j=1}^{N} p_j K_{a_j} : N \in \mathbb{N} \text{ and } p_j \in \mathcal{P}_n, a_j \in \mathbb{C}^n \text{ for } j = 1, \ldots, N \right\}.
\]

recall that \( K_a \) denotes the reproducing kernel at \( a \in \mathbb{C}^n \) for the space \( H^2(\mathbb{C}^n) \). We will characterize solutions of (3.2) in \( \mathcal{A}_n \). In what follows we use the notation

\[
K_{a,b} := K_a K_b
\]

for \( a, b \in \mathbb{C}^n \). Also, we put

\[
f^*(z) := \overline{f(z)}
\]

for \( f \in Hol(\mathbb{C}^n) \).

**Lemma 3.3.** The equality

\[
\mathcal{B}[pqK_{a,b}](z) = e^{b\pi} K_{a,b}(z) q^*(\partial_z + \overline{z} + \overline{a}) p(z + b), \quad z \in \mathbb{C}^n,
\]

holds for \( a, b \in \mathbb{C}^n \) and \( p, q \in \mathcal{P}_n \).

**Proof.** Fix \( a, b \in \mathbb{C}^n \) and \( p, q \in \mathcal{P}_n \). Using the representation \( q(w) =: \sum_{\alpha} c_{\alpha} w^{\alpha} \), we have

\[
\mathcal{B}[pqK_{a,b}](z) = e^{-|z|^2} \sum_{\alpha} \overline{c}_{\alpha} I_{\alpha}(z)
\]

where

\[
I_{\alpha} := \int_{\mathbb{C}^n} p(w) \overline{w}^{\alpha} \exp \left\{ w \cdot (\overline{a} + \overline{z}) + (b + z) \cdot \overline{w} \right\} d\mu(w).
\]

For each \( \alpha \), note from the reproducing property that

\[
I_{\alpha}(z) = \int_{\mathbb{C}^n} p(w) \partial_{z}^{\alpha} \left[ \exp \left\{ w \cdot (\overline{a} + \overline{z}) + (z + b) \cdot \overline{w} \right\} \right] d\mu(w)
\]

\[
= \partial_{\alpha} \int_{\mathbb{C}^n} p(w) e^{w \cdot (\overline{a} + \overline{z})} K_{z+b}(w) d\mu(w)
\]

\[
= \partial_{\alpha} \left[ p(z + b) e^{(z+b) \cdot (\overline{a} + \overline{z})} \right]
\]

and, in addition, that the relation

\[
\partial_{\alpha} \left[ p(z + b) e^{(z+b) \cdot (\overline{a} + \overline{z})} \right] = e^{(z+b) \cdot (\overline{a} + \overline{z})} \partial_{\alpha} (\overline{a} + \overline{z} + a) p(z + b)
\]

\[
= e^{b \pi |z|^2} K_{a,b}(z) (\overline{a} + \overline{z} + a) \alpha p(z + b)
\]

holds. Combining these observations, we conclude the lemma. \( \square \)

We remark that the Berezin transform of a polynomial is again a polynomial by the case \( a = b = 0 \) of Lemma 3.3. In fact the Berezin transform is an isomorphism on the space of all polynomials in \( z \) and \( \overline{z} \) preserving the degree; see [4].

**Lemma 3.4.** For \( a, b \in \mathbb{C}^n \) and \( p, q \in \mathcal{P}_n \setminus \{0\} \), assume

\[
\mathcal{B}[pqK_{a,b}] = pqK_{a,b}.
\]

Then \( a \cdot \overline{b} \in 2\pi i \mathbb{Z} \) and the following statements hold:

(a) If \( a \neq 0 \), then \( q \) is constant.
(b) If \( b \neq 0 \), then \( p \) is constant.

**Proof.** By Lemmas 3.3 and 2.9 we have

\[
p(z)q^*(w) = e^{b \pi} q^* (\partial_z + w + \bar{w}) p(b + z)
\]

for all \( z, w \in \mathbb{C}^n \). This, with \( z \) fixed, can be regarded as a \( w \)-polynomial identity. Comparing the coefficients of the highest \( w \)-degree terms, we obtain a \( z \)-polynomial identity

\[
p(z) = e^{b \pi} p(z + b).
\]

Since \( p \) is not the zero polynomial, it follows that \( e^{b \pi} = 1 \), or equivalently, \( a \cdot \bar{b} \in 2\pi i \mathbb{Z} \). Moreover, if \( b \neq 0 \), then \( p \) is constant by \( b \)-periodicity. This shows (b). Note that (a) is the same statement as (b) via the complex conjugation. \( \square \)

Note that a function \( f \in \mathcal{A}_n \) admits a **canonical** representation

\[
f = \sum_{j=0}^{J} p_j K_{a_j}
\]

where \( p_0 \in \mathcal{P}_n \), \( p_j \in \mathcal{P}_n \setminus \{0\} \) for each \( j \geq 1 \) and \( \{a_0 = 0, \ldots, a_J\} \) is a collection of distinct points in \( \mathbb{C}^n \). Consider another \( g \in \mathcal{A}_n \) with canonical representation

\[
g = \sum_{\ell=0}^{L} q_{\ell} K_{b_{\ell}}.
\]

The following characterization shows that the action of the Berezin transform on \( \mathcal{A}_n \) is quite rigid.

**Theorem 3.5.** For functions \( f, g \in \mathcal{A}_n \) with canonical representations as in (3.5) and (3.6), the following statements are equivalent:

(a) \( \mathcal{B}[f\overline{g}] = f\overline{g} \).

(b) \( \mathcal{B}[p_j \overline{q}_{\ell} K_{a_j,b_{\ell}}] = p_j \overline{q}_{\ell} K_{a_j,b_{\ell}} \) for each \( j \) and \( \ell \).

If, in addition, neither \( f \) nor \( g \) is a polynomial, then either condition of the above is also equivalent to

(c) Both \( p_j, q_{\ell} \) are constants and \( a_j \overline{b}_{\ell} \in 2\pi i \mathbb{Z} \) for each \( j \) and \( \ell \).

**Proof.** Using the notation

\[
R_{a,b,p,q} = R_{a,b,p,q}(z, \bar{z}) := e^{b \pi} q^* (\partial_z + \bar{z} + \bar{\bar{w}}) p(b + z),
\]

we have by Lemma 3.3

\[
\mathcal{B}[f\overline{g}] = \sum_{j,\ell} \mathcal{B}[p_j \overline{q}_{\ell} K_{a_j,b_{\ell}}] = \sum_{j,\ell} R_{a_j,b_{\ell},p_j,q_{\ell}} K_{a_j,b_{\ell}}.
\]

So, assuming (a), we have

\[
\sum_{j,\ell} \left[ R_{a_j,b_{\ell},p_j,q_{\ell}} - p_j \overline{q}_{\ell} \right] K_{a_j,b_{\ell}} = 0.
\]

Recall that each \( R_{a_j,b_{\ell},p_j,q_{\ell}} \) is a polynomial in \( z \) and \( \bar{z} \). Note that the functions \( K_{a_j,b_{\ell}} \) form a linearly independent set over the polynomials, because \( (a_j, b_{\ell})'s \) are all distinct. It follows from the above that \( R_{a_j,b_{\ell},p_j,q_{\ell}} = p_j \overline{q}_{\ell} \) for each \( j \) and \( \ell \). So, (b) holds. The implication (b) \( \implies \) (a) is clear. Finally, when neither \( f \) nor \( g \) is a polynomial, the equivalence (b) \( \iff \) (c) holds by Lemmas 3.3 and 3.4. \( \square \)
For any $a, b \in \mathbb{C}^n$, note that $K_{a,b}$ is a $\Delta$-eigenfunction with eigenvalue $\bar{a} \cdot b$. Thus Theorem 3.5 suggests a special role played by $\Delta$-eigenfunctions with eigenvalues in $2\pi i \mathbb{Z}$. Motivated by this observation, we now proceed to the investigation into the $B$-fixed points in question, i.e., the solutions of (3.1), by means of such eigenfunctions.

**Proposition 3.6.** Given $f, g \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n)$, the series

$$S(\lambda, z) := \sum_{\alpha \in \mathbb{N}_0^n} \frac{\lambda^{||\alpha||}}{\alpha!} f^{(\alpha)}(z) g^{(\alpha)}(z), \quad (\lambda, z) \in \mathbb{C} \times \mathbb{C}^n,$$

converges absolutely. Moreover, $S(\lambda, \cdot) \in \text{Sym}(\mathbb{C}^n)$ for each $\lambda \in \mathbb{C}$.

**Proof.** Let $f, g \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n)$. Given $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, put $t := |\lambda| > 0$. Since $f$ is fixed by the heat transform, we have

$$f(z) = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(w) \exp \left\{ -\frac{|z-w|^2}{t} \right\} dv(w)$$

for $z \in \mathbb{C}^n$. Using this, we now estimate the growth rate of derivatives of $f$. Given $\alpha \in \mathbb{N}_0^n$, differentiating under the integral sign, we obtain

$$f^{(\alpha)}(z) = \frac{1}{\pi^n t^{n+||\alpha||}} \int_{\mathbb{C}^n} f(w)(\overline{w}-\overline{z})^\alpha \exp \left\{ -\frac{|z-w|^2}{t} \right\} dv(w)$$

$$= \frac{1}{\pi^n t^{n+||\alpha||}} \int_{\mathbb{C}^n} f(\xi + z) \xi^\alpha e^{-|\xi|^2/t} dv(\xi).$$

Thus, setting

$$I_f(\alpha, z) := \int_{\mathbb{C}^n} |f(\xi + z)||\xi^\alpha| e^{-|\xi|^2/t} dv(\xi),$$

we have $|f^{(\alpha)}(z)| \leq (\pi^n t^{n+||\alpha||})^{-1} I_f(\alpha, z)$. The same inequality holds with $g$ in place of $f$. We thus obtain

$$\frac{t^{||\alpha||} |f^{(\alpha)}(z) g^{(\alpha)}(z)|}{\alpha!} \leq \frac{1}{(t\pi)^{2n}} \frac{I_f(\alpha, z) I_g(\alpha, z)}{\alpha! t^{||\alpha||}}$$

for all $z \in \mathbb{C}^n$.

We now estimate the right hand side of (3.9). Let $0 < \varepsilon < \frac{1}{2t}$. Since

$$|f(\xi + z)| \leq \|f\|_{\mathcal{D}_e} e^{\varepsilon |\xi + z|^2} \leq \|f\|_{\mathcal{D}_e} e^{2\varepsilon (|\xi|^2 + |z|^2)},$$

it follows that

$$|I_f(\alpha, z)| \leq \|f\|_{\mathcal{D}_e} e^{2\varepsilon |z|^2} \prod_{j=1}^n |\zeta^{|\alpha_j|} \exp \{ (2\varepsilon - t^{-1}) |\zeta|^2 \} dv_1(\zeta)$$

$$= (2\pi)^n \|f\|_{\mathcal{D}_e} e^{2\varepsilon |z|^2} \prod_{j=1}^n \int_0^\infty r^{\alpha_j+1} e^{-(t^{-1}-2\varepsilon)r^2} dr$$

$$=: (*).$$

Applying the identity

$$\int_0^\infty r^{2b-1} e^{-ar^2} dr = \frac{\Gamma(b)}{2a^b}$$
valid for positive numbers $a$ and $b$, we have
\[(*) = \frac{\pi^n t^n + \frac{|a|}{2} m}{(1 - 2\varepsilon t)^{n+|a|}} \prod_{j=1}^n \Gamma \left( \frac{\alpha_j}{2} + 1 \right).\]

The same estimate holds with $g$ in place of $f$. It follows that the right hand side of (3.11) is less than or equal to
\[(3.10) \quad \frac{\|f\|_{D_2} \|g\|_{D_2} e^{4|z|\varepsilon^2}}{(1 - 2\varepsilon t)^{2n}} \cdot \frac{1}{(1 - 2\varepsilon t)^{|a|}} \prod_{j=1}^n \frac{\Gamma^2 \left( \frac{\alpha_j}{2} + 1 \right)}{\alpha_j!}.\]

In conjunction with the product of quotients of Gamma functions above, we note via Stirling’s formula the estimate
\[\frac{\Gamma^2 \left( \frac{m}{2} + 1 \right)}{m!} = \frac{\Gamma^2 \left( \frac{m}{2} + 1 \right)}{\Gamma(m + 1)} \leq C \sqrt{m + 1} \frac{2^m}{2m}, \quad m = 0, 1, 2, \ldots,
\]
with an absolute constant $C > 0$. Applying this to each factor of the product in (3.10), we obtain
\[(3.11) \quad \frac{1}{(t\pi)^{2n}} \frac{I_f(\alpha, z) I_g(\alpha, z)}{\alpha^{|\alpha|}} \leq C_\varepsilon(z) \frac{(1 + |\alpha|)^{n/2}}{2^{|\alpha|}(1 - 2t^2\varepsilon)^{|\alpha|}}\]

where $C_\varepsilon(z) := C^n \|f\|_{D_2} \|g\|_{D_2} e^{4|z|\varepsilon^2} (1 - 2t\varepsilon)^{-2n}$. Note that $\varepsilon$ can be chosen arbitrarily small. Also, given an integer $m \geq 0$, the number of multi-indices $\alpha$ with $|\alpha| = m$ is $(n+m-1)_m$, which is comparable to $m^{n-1}$. So, choosing sufficiently small $\varepsilon > 0$, say $\varepsilon \leq \frac{1}{\sqrt{t}}$, the sequence on the right hand side of (3.11) is summable over $\alpha \in \mathbb{N}_0^n$ and the assertion follows from (3.9).

The formal expression of the heat transform
\[\mathcal{B}_t = \sum_{j=0}^{+\infty} \frac{(t\Delta)^j}{j!} = e^{t\Delta}\]
is well known; see, for example, [15]. The next lemma shows that this formal expression is actually the case for functions under consideration. Note that the series in the next lemma converges according to Proposition 3.6.

**Lemma 3.7.** Given $f, g \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n)$, the series expansion
\[(3.12) \quad \mathcal{B}_t[f, g](z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{t^{|\alpha|}}{\alpha!} f^{(\alpha)}(z) \overline{g^{(\alpha)}(z)}\]
is valid for all $t > 0$ and $z \in \mathbb{C}^n$.

**Proof.** Let $f, g \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. Note from (3.8)
\[f^{(\alpha)}(z) \overline{g^{(\alpha)}(z)} = \frac{1}{i^{2|\alpha|}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} f(\xi + z) \overline{g(w + z)} w^\alpha \overline{\xi^\alpha} \, d\mu_\xi(\xi) \, d\mu_\mu(w)\]
for each $\alpha \in \mathbb{N}_0^n$ and $t > 0$. Multiplying by $\frac{t^{|\alpha|}}{\alpha!}$ both sides of the above equality and then taking the sum over all $\alpha \in \mathbb{N}_0^n$ under the integral sign (justified by the growth estimate
Thus, setting \( t \) which completes the proof.

Recall that \( e^{\frac{z}{\tau}} \) is the reproducing kernel for \( H^2_\tau(\mathbb{C}^n) \). So, the double integral of the above reduces to

\[
\int_{\mathbb{C}^n} f(w + z)g(w + z) d\mu_t(w) = B_t[f\overline{g}](z),
\]

which completes the proof. \( \square \)

Given a \( B \)-fixed point \( u \in \text{Sym}(\mathbb{C}^n) \), recall that \( \Lambda_u(\cdot, z) \) (with \( z \in \mathbb{C}^n \) fixed) extends to a 1-periodic entire function on \( \mathbb{C} \); see (2.10). In the next lemma \( \Lambda_u \) denotes such extension on \( \mathbb{C}^{n+1} \). Recall that \( \mathcal{E}(\mathbb{C}^n) \) denotes the class of all functions over \( \mathbb{C}^n \) of exponential type.

**Lemma 3.8.** If \( u \in \mathcal{E}(\mathbb{C}^n) \) and \( B[u] = u \), then \( \Lambda_u \in \mathcal{E}(\mathbb{C}^{n+1}) \).

**Proof.** Let \( u \in \mathcal{E}(\mathbb{C}^n) \) and assume \( B[u] = u \). Let \( \lambda \in \mathbb{C} \) be given. We may assume \( \text{Re} \lambda > 0 \) by periodicity. Since \( \text{Re} \lambda > 0 \), we have

\[
\Lambda_u(\lambda, z) = \frac{1}{(\pi \lambda)^n} \int_{\mathbb{C}^n} u(z + w) e^{-\frac{|w|^2}{\lambda}} dw.
\]

Thus, setting \( t := \text{Re} \lambda > 0 \) and choosing \( s > 0 \) such that

\[
\| u \|_{\mathcal{E}_s} := \sup_{z \in \mathbb{C}^n} |u(z)| e^{-s|z|} < \infty,
\]

we have

\[
|\lambda|^n e^{-s|z|} |\Lambda_u(\lambda, z)| \leq \frac{\| u \|_{\mathcal{E}_s}}{\pi^n} \int_{\mathbb{C}^n} \exp \left\{ s|w| - \frac{t}{|\lambda|^2}|w|^2 \right\} dw = c_n \| u \|_{\mathcal{E}_s} \int_0^{\infty} r^{2n-1} \exp \left\{ sr - \frac{tr^2}{|\lambda|^2} \right\} dr,
\]

where \( c_n \) is a constant depending only on the dimension \( n \). Meanwhile, the integral above is dominated by some constant, depending on \( a \), times

\[
\int_{-\infty}^{\infty} \exp \left\{ 2sr - \frac{tr^2}{|\lambda|^2} \right\} dr = |\lambda| \sqrt{\frac{\pi}{t}} \exp \left\{ \frac{s^2|\lambda|^2}{t} \right\}.
\]

Now, further assuming by 1-periodicity that \( t \geq \max(|\text{Im} \lambda|, 1) \), we obtain

\[
|\Lambda_u(\lambda, z)| \leq \frac{C \| u \|_{\mathcal{E}_s}}{\sqrt{t}|\lambda|^{n-1}} \exp \left\{ s|z| + \frac{s^2|\lambda|^2}{t} \right\} \leq C \| u \|_{\mathcal{E}_s} e^{s|z|+2s^2|\lambda|}
\]

with a constant \( C = C(n, s) > 0 \). This completes the proof. \( \square \)

**Proposition 3.9.** Given a positive integer \( N \), let \( f_\ell, g_\ell \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n) \) for \( \ell = 1, \ldots, N \). Put

\[
u := \sum_{\ell=1}^{N} f_\ell \overline{g_\ell}
\]

(3.13)
and assume $\mathcal{B}[u] = u$. Then there is a sequence of functions \( \{\varphi_m\}_{m \in \mathbb{Z}} \subset \text{Sym}_2(\mathbb{C}^n) \) with the following properties:

(a) For \( z \in \mathbb{C}^n \) and \( \lambda \in \mathbb{C} \)

\[
\Lambda_u(\lambda, z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\lambda^{\vert \alpha \vert}}{\alpha!} \left[ \sum_{\ell=1}^{N} f^{(\alpha)}_{\ell}(z) g^{(\alpha)}_{\ell}(z) \right] = \sum_{m=-\infty}^{\infty} \varphi_m(z)e^{2\pi im\lambda},
\]

where both series converge absolutely, and moreover, for each fixed \( z \), the series on the right hand side converges uniformly on each horizontal strip in \( \mathbb{C} \).

(b) Each \( \varphi_m \) admits representations

\[
\varphi_m(z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\gamma(|\alpha|, m)}{\alpha!} \left[ \sum_{\ell=1}^{N} f^{(\alpha)}_{\ell}(z) g^{(\alpha)}_{\ell}(z) \right] = \int_0^1 B_t \left[ \sum_{\ell=1}^{N} f^{(\alpha)}_{\ell}(z) g^{(\alpha)}_{\ell}(z) \right] e^{-2\pi imt} dt
\]

where coefficients \( \gamma(|\alpha|, m) \) are given by

\[
\gamma(|\alpha|, m) := \int_0^1 t^{\vert \alpha \vert} e^{-2\pi imt} dt.
\]

(c) Given \( 0 < \varepsilon \leq \frac{1}{2} \) and \( j \in \mathbb{N} \),

\[
\sup_{m \in \mathbb{Z}} |m|^j \|\varphi_m\|_{\mathcal{D}_{2\varepsilon}} < \infty.
\]

(d) If, in addition, \( u \in \mathcal{E}(\mathbb{C}^n) \), then \( \varphi_m \neq 0 \) only for finitely many \( m \)'s.

Proof. Since \( u \) is a \( \mathcal{B} \)-fixed point by assumption, the function \( \Lambda_u(\cdot, z) \) with \( z \) fixed extends to a 1-periodic entire function on \( \mathbb{C} \), still denoted by \( \Lambda_u(\cdot, z) \). Thus, by Proposition 2.6, \( \Lambda_u(\cdot, z) \) can be expressed in the form

\[
\Lambda_u(\lambda, z) = \sum_{m=-\infty}^{\infty} \varphi_m(z)e^{2\pi im\lambda}
\]

with coefficients \( \varphi_m(z) \) depending on \( z \). Now, (a) is a consequence of Lemma 3.7. When \( \lambda = t \in [0, 1] \) (with \( z \) fixed), note that the right hand side of (3.14) is the Fourier series of \( \Lambda_u(t, z) \). Thus (b) holds by (a). Note that due to Lemma 2.1 functions of the form (3.13) define elements in \( \text{Sym}_2(\mathbb{C}^n) \). According to Lemma 2.2 it thus follows from the integral expression in (b) that functions \( \varphi_m \) are elements in \( \text{Sym}_2(\mathbb{C}^n) \) for all \( m \in \mathbb{Z} \).

Now, we prove (c). Fix \( j \in \mathbb{N} \). We have by periodicity of \( \Lambda_u(\cdot, z) \) for any \( z \in \mathbb{C}^n \) and \( m \):

\[
\varphi_m(z) = \int_0^1 \Lambda_u(t, z)e^{-2\pi imt} dt = \frac{(-2\pi im)^j}{(2\pi im)^j} \int_0^1 \Lambda_u(t, z) \partial^j_t [e^{-2\pi imt}] dt
\]

(3.15)
In conjunction with this, recall that $\Lambda_u(t, z)$ is a solution of the heat equation (2.7) and that the Laplacian $\Delta$ commutes with $B_t$ on $\text{Sym}_2(\mathbb{C}^n)$. Therefore, setting

$$h_j := \Delta^j u = \sum_{|\alpha|=j} \frac{j!}{\alpha!} \sum_{\ell=1}^N f^{(\alpha)}_{\ell} g^{(\alpha)}_{\ell},$$

we obtain by an inductive argument

$$\partial^j_t \Lambda_u(t, z) = \Delta^j \Lambda_u(t, z) = B_t[h_j](z);$$

(3.16)

note $u, \Delta u, \ldots, \Delta^{i-1} u \in \text{Sym}_2(\mathbb{C}^n)$ by Lemma 2.1. Let $0 < \varepsilon \leq \frac{1}{2}$. For any $t \in (0, 1]$, we have by Lemma 2.2

$$\|\partial^j_t \Lambda_u(t, \cdot)\|_{D_{2(t)}} = \|B_t[h_j]\|_{D_{2(t)}} \leq 2^n \|h_j\|_{D_t}.$$

This, together with the identity (3.15), yields

$$|m|^j \|\varphi_m\|_{D_{2(t)}} \leq \frac{1}{(2\pi)^j} \int_0^1 \|\partial^j_t \Lambda_u(t, \cdot)\|_{D_{2(t)}} dt \leq \frac{\|h_j\|_{D_t}}{2^n (2\pi)^j}$$

and thus we conclude (c), as asserted.

Finally, we prove (d). Assume $u \in \mathcal{E}(\mathbb{C}^n)$. We then have $\Lambda_u \in \mathcal{E}(\mathbb{C}^{n+1})$ by Lemma 3.8. So, there is some $s > 0$ such that $|\Lambda_u(\lambda, z)| e^{-s(|\lambda| + |z|)}$ is bounded on $\mathbb{C}^{n+1}$. It follows that $\varphi_m$ disappears for all $m$ with $|m| > s$ by Lemma 2.5. The proof is complete. \hfill \Box

Let $\varphi$ be a $\Delta$-eigenfunction with eigenvalue $\lambda$. Then the function $\Phi(t, z) := e^{\lambda t} \varphi(z)$ clearly solves the initial value problem for the heat equation:

$$(\Delta_z - \partial_t) \Phi(t, z) = 0, \quad \Phi(0, z) = \varphi(z).$$

If, in addition, $\varphi \in \text{Sym}_2(\mathbb{C}^n)$, then the next lemma shows that the heat transform $B_t[\varphi](z)$ turns out to coincide with $\Phi(t, z)$.

**Lemma 3.10.** Let $\varphi \in \text{Sym}_2(\mathbb{C}^n)$ be a $\Delta$-eigenfunction with eigenvalue $\lambda \in \mathbb{C}$. Then

$$B_t[\varphi] = e^{\lambda t} \varphi$$

for all $t > 0$. In particular, $B[\varphi] = \varphi$ whenever $\lambda \in 2\pi i \mathbb{Z}$.

**Proof.** Let $z \in \mathbb{C}^n$. As was noticed before, the function $\Phi(t, z) := B_t[\varphi](z)$ with $z$ fixed extends to a holomorphic function on the right half-plane. Since $\Delta \varphi = \lambda \varphi$, we have as in (3.16)

$$\partial_t^j B_t[\varphi](z) = B_t[\Delta^j \varphi](z) = \lambda^j B_t[\varphi](z)$$

for $t > 0$ and each nonnegative integer $j$. This shows that the $j$-th Taylor coefficient at $s > 0$ of the function $\Phi(\cdot, z)$ is $\frac{\lambda^j}{j!} B_s[\varphi](z)$. Accordingly, we have

$$B_t[\varphi](z) = \sum_{j=0}^{\infty} \frac{(t-s)^j \lambda^j}{j!} B_s[\varphi](z) = e^{\lambda(t-s)} B_s[\varphi](z)$$

(3.17)

for any $t, s > 0$.

Meanwhile, by an elementary change of variables in the definition of heat transform (2.5), we have

$$B_s[\varphi](z) = \int_{\mathbb{C}^n} \varphi(z + \sqrt{sw}) d\mu(w)$$
for \( s > 0 \). In the integral on the right hand side of the above, note that the integrand tends to \( \varphi(z) \) as \( s \downarrow 0 \). In addition, since \( \varphi \in \text{Sym}(\mathbb{C}^n) \), we have

\[
|\varphi(z + \sqrt{s}w)| \leq \|\varphi\|_{D_{1/4}} \exp \left\{ \frac{|z + \sqrt{s}w|^2}{4} \right\} \\
\leq \|\varphi\|_{D_{1/4}} \exp \left\{ \frac{|z|^2 + |w|^2}{2} \right\}
\]

for \( 0 < s \leq 1 \). Since the function above is \( \mu \)-integrable as a function of \( w \), we deduce by the Lebesgue dominated convergence theorem that \( \lim_{s \downarrow 0} B_s[\varphi](z) = \varphi(z) \). Consequently, taking the limit \( s \downarrow 0 \) in (3.17), we conclude the lemma. \( \square \)

We are now ready to prove the following eigenfunction-series characterization for solutions of (3.1) in the symbol space. In what follows, we denote by \( X_m(\mathbb{C}^n) \) the \( \Delta \)-eigenspace in \( \text{Sym}_2(\mathbb{C}^n) \) with eigenvalue \( 2\pi mi \), i.e.,

\[
X_m(\mathbb{C}^n) := \{ u \in \text{Sym}_2(\mathbb{C}^n) : \Delta u = 2\pi miu \}
\]

for each integer \( m \).

Theorem 3.11. Given a positive integer \( N \), let \( f_\ell, g_\ell \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n) \) for \( \ell = 1, \ldots, N \) and put

\[
u := \sum_{\ell=1}^{N} f_\ell \overline{g_\ell}.
\]

Then the following statements are equivalent:

(a) \( B[u] = u \).

(b) There are functions \( \varphi_m \in X_m(\mathbb{C}^n) \) for \( m \in \mathbb{Z} \) such that

\[
u = \sum_{m=-\infty}^{\infty} \varphi_m,
\]

where the series converges in the Fréchet topology of \( \text{Sym}(\mathbb{C}^n) \).

If, in addition, \( u \in \mathcal{E}(\mathbb{C}^n) \), then the series in (b) reduces to a finite sum.

Proof. First, assume (b). Recall from Lemma 2.2 that the Berezin transform \( B \) acts continuously on \( \text{Sym}(\mathbb{C}^n) \). Therefore we obtain by Lemma 3.10

\[
B[u] = \sum_{m=-\infty}^{\infty} B[\varphi_m] = \sum_{m=-\infty}^{\infty} \varphi_m = u
\]

so that (a) holds.

Conversely, assume (a). By Proposition 3.9(a) there are functions \( \varphi_m \in \text{Sym}_2(\mathbb{C}^n) \) such that

\[
\sum_{\alpha \in \mathbb{N}_0^n} \frac{j_{\alpha}}{\alpha!} \left[ \sum_{\ell=1}^{N} f_\ell^{(\alpha)} \overline{g_\ell^{(\alpha)}} \right] = \sum_{m=-\infty}^{\infty} \varphi_m e^{2\pi imt}
\]

for \( t \geq 0 \); recall that both series converge absolutely. Given \( 0 < \varepsilon \leq \frac{1}{2} \), we have by Proposition 3.9(c)

\[
\|\varphi_m\|_{D_{2\varepsilon}} \leq \frac{C_\varepsilon}{|m|^2}, \quad m = \pm 1, \pm 2, \ldots,
\]
for some constant $C_\varepsilon > 0$ independent of $m$. This implies that the series on the right hand side of (3.18) actually converges in the Fréchet topology of $\text{Sym}(\mathbb{C}^n)$ (uniformly in $t$ real). In particular, taking $t = 0$, we have

$$u = \sum_{m=-\infty}^{\infty} \varphi_m$$

with the series convergent in the Fréchet topology of $\text{Sym}(\mathbb{C}^n)$.

Recall that the function $\Lambda_u(t, z)$ satisfies the heat equation (2.7). In addition, since $u$ is fixed by the Berezin transform by assumption, $\Lambda_u(t, z)$ is 1-periodic as a function of $t$. Also, by Proposition 3.9(b), we have the integral representation

$$\varphi_m(z) = \int_0^1 \Lambda_u(t, z) e^{-2\pi imt} dt$$

for each $m$. This shows that each $\varphi_m$ is real analytic. In addition, applying the Laplacian under the integral sign, we have

$$\Delta \varphi_m(z) = \int_0^1 \Delta_z \Lambda_u(t, z) e^{-2\pi imt} dt$$

$$= \int_0^1 \partial_t \Lambda_u(t, z) e^{-2\pi imt} dt$$

$$= -\int_0^1 \Lambda_u(t, z) \partial_t[e^{-2\pi imt}] dt$$

$$= 2\pi im \varphi_m(z)$$

for each $m$. So, we conclude that (b) holds.

When $u \in \mathcal{E}(\mathbb{C}^n)$, note that the series on the right hand side of (3.18) and thus that of (3.19) is actually a finite sum by Proposition 3.9(d). This completes the proof.

We now observe a consequence of Proposition 3.9.

**Corollary 3.12.** Given a positive integer $N$, let $f_\ell, g_\ell \in \text{Sym}(\mathbb{C}^n) \cap \text{Hol}(\mathbb{C}^n)$ for $\ell = 1, \ldots, N$ and assume that $f_\ell$ or $g_\ell$ is a polynomial for each $\ell$. Then the following statements are equivalent:

(a) $B \left[ \sum_{\ell=1}^{N} f_\ell g_\ell \right] = \sum_{\ell=1}^{N} f_\ell g_\ell$.

(b) $\sum_{\ell=1}^{N} f_\ell g_\ell$ is harmonic.

**Proof.** The implication (b) $\implies$ (a) is clear, because harmonic functions in the symbol space are fixed by the Berezin transform. Conversely, assume (a) and put $u := \sum_{\ell=1}^{N} f_\ell \overline{g_\ell}$. Since $f_\ell$ or $g_\ell$ is a polynomial for each $\ell$ by assumption, we see from Proposition 3.9(a) that $\Lambda_u(\cdot, z)$ with fixed $z \in \mathbb{C}^n$ is a 1-periodic polynomial and thus is constant. It follows that $\sum_{\ell=1}^{N} f_\ell(\alpha) \overline{g_\ell(\alpha)} = 0$ for any multi-index $\alpha \neq 0$. Now, the case $|\alpha| = 1$ implies (b).
Remark 3.13. When \( n = 1 \), condition (b) of Corollary 3.12 can be described more explicitly
by the condition
\[
\sum_{\ell=1}^N [f_\ell - f(0)] [g_\ell - g(0)] = 0;
\]
see [11, Theorem 3.3]. In particular, when both pairs \( \{f_1, f_2\} \) and \( \{g_1, g_2\} \) contain
a nonconstant function, it is elementary to check that \( f_1 \overline{g_1} + f_2 \overline{g_2} \) is harmonic
if and only if
\[
(cf_1 + df_2 = c_1 \quad \text{and} \quad \overline{dg_1} - \overline{cg_2} = c_2)
\]
for some constants \( c, d, c_1, c_2 \) with \((c, d) \neq (0, 0)\). We also observe a consequence of
Theorem 3.11. To see sequences satisfying property (b) of the next corollary, one may take
\( a_j := 2\pi i m_j \zeta \) and \( b_\ell := n_\ell \eta \) where \( \zeta, \eta \in \mathbb{C}^n \) with \( \zeta \cdot \overline{\eta} = 1 \)
and \( \{m_j\}, \{n_\ell\} \) are sequences of distinct integers. Also, when the series
in (3.22) are finite sums, note that the next corollary is contained in Theorem 3.5.

Corollary 3.14. Let \( \{a_j\}, \{b_\ell\} \) be sequences of points in \( \mathbb{C}^n \setminus \{0\} \) such that
\((a_j, b_\ell)\)'s are all distinct and let \( \{c_j\}, \{d_\ell\} \) be sequences of complex numbers such that
\[
\sum_{j=1}^\infty \left( |c_j| e^{t|a_j|^2} + |d_\ell| e^{t|b_\ell|^2} \right) < \infty
\]
for any \( t > 0 \). Put
\[
f := \sum_{j=1}^\infty c_j K_{a_j} \quad \text{and} \quad g := \sum_{\ell=1}^\infty d_\ell K_{b_\ell}.
\]
Then the following statements are equivalent:
(a) \( \mathcal{B}[fg] = fg \).
(b) For each \( j \) and \( \ell \), either \( c_j d_\ell = 0 \) or \( a_j \cdot \overline{b_\ell} \in 2\pi i \mathbb{Z} \).

Proof. Note \( \|K_a\|_t = \|K_{a_t}\|_t = e^{t|a|^2/2} \). Thus we have by (3.21)
\[
\sum_{j=1}^\infty |c_j| \|K_{a_j}\|_t < \infty \quad \text{and} \quad \sum_{\ell=1}^\infty |d_\ell| \|K_{b_\ell}\|_t < \infty
\]
for any \( t > 0 \). Thus we see from (2.2) that the defining series in (3.22) for \( f \) and \( g \) converge in the Fréchet topology of the symbol space. Note
\[
\|K_a K_b\|_t = \|K_{a+b}\|_t = \exp \left\{ \frac{t|a + b|^2}{2} \right\} \leq \exp \left\{ t(|a|^2 + |b|^2) \right\}
\]
for \( a, b \in \mathbb{C}^n \) and \( t > 0 \). Using this inequality, we also see from (3.21) and (2.2) that the series
\[
f g = \sum_{j,\ell} c_j \overline{d_\ell} K_{a_j} K_{b_\ell} = \sum_{j,\ell} c_j \overline{d_\ell} K_{a_j, b_\ell}
\]
converges in the Fréchet topology of the symbol space. Note that each \( K_{a_j, b_\ell} \) is a \( \Delta \)-eigenfunction with eigenvalue \( \overline{a_j} \cdot b_\ell \). Also, note that any finite sub-collection of \( \{K_{a_j, b_\ell}\}_{j,\ell \in \mathbb{Z}} \) is linearly independent, because \((a_j, b_\ell)\)'s are all distinct by assumption. The assertion now follows from Theorem 3.11.

We now close this section with a remark on the range of the Berezin transform.
Remark 3.15. The Berezin transform $B_D$ associated with the Bergman space over the unit disk $D$ of the complex plane is defined in a similar way by means of the normalized reproducing kernels; we refer to [28] for the precise definition and related facts. In [1, 22] solutions to an equation of the following form are completely determined:

$$B_D[u] = \sum_{\ell=1}^{N} f_{\ell} \overline{g_{\ell}}$$

where $f_{\ell}, g_{\ell} \in Hol(D)$ and $u \in L^{1}(D)$. It turns out that this type of equation can be solved only when the right hand side is of very restricted form. Here, we want to point out that the analogous equation for the Berezin transform $B$ associated with the Fock space can be solved rather easily for a large class of entire functions in the symbol space.

Given $f_{\ell}, g_{\ell} \in \text{Sym}(\mathbb{C}^{n})$ for $\ell = 1, \ldots, N$, consider the equation

$$\sum_{\ell=1}^{N} f_{\ell}(z) \overline{g_{\ell}(z)} = B[u] \quad \text{with} \quad u \in L^{2}(\mathbb{C}^{n}, d\mu),$$

or more explicitly,

$$\sum_{\ell=1}^{N} f_{\ell}(z) \overline{g_{\ell}(z)} = \frac{e^{-|z|^{2}}}{\pi^{n}} \int_{\mathbb{C}^{n}} u(\xi) \exp \left\{ z \cdot \overline{\xi} + z \cdot \xi - |\xi|^{2} \right\} d\nu(\xi).$$

By Lemma 2.9 we have

$$Q(z, w) = \sum_{\ell=1}^{N} f_{\ell}(z) \overline{g_{\ell}^{*}(w)} = \frac{e^{-z \cdot w}}{\pi^{n}} \int_{\mathbb{C}^{n}} u(\xi) \exp \left\{ z \cdot \overline{\xi} + w \cdot \xi - |\xi|^{2} \right\} d\nu(\xi)$$

for $z, w \in \mathbb{C}^{n}$. By Lemma 2.9 again, this is equivalent to

$$e^{-|z|^{2}} Q \left( \frac{iz}{2}, \frac{iz}{2} \right) = \frac{1}{\pi^{n}} \int_{\mathbb{C}^{n}} u(\xi) e^{-|\xi|^{2}} e^{i\text{Re}(z \xi)} d\nu(\xi)$$

$$=: \mathcal{F}(ue^{-|\cdot|^{2}})(z),$$

where $\mathcal{F}$ stands for the Fourier transform. Note that the function $z \mapsto Q \left( \frac{iz}{2}, \frac{iz}{2} \right)$ belongs to the symbol space. Applying the inverse Fourier transform $\mathcal{F}^{-1}$ on $L^{2}(\mathbb{C}^{n}, d\nu)$, we obtain

$$u(\xi) = e^{i|\xi|^{2}} \mathcal{F}^{-1} \left[ e^{-\frac{|\xi|^{2}}{4}} Q \left( \frac{iz}{2}, \frac{iz}{2} \right) \right](\xi), \quad \xi \in \mathbb{C}^{n}.$$ 

So, the function $u$ above is the unique solution of (3.23).

4. (Semi-)Commuting Toeplitz operators: One-dimensional case

In this section our discussion is restricted to the one-dimensional case $n = 1$; the multi-dimensional case is left open.

We aim to characterize (semi-)commuting Toeplitz operators with symbols of exponential type as in (2.14). We note that the holomorphic part and co-holomorphic part of a pluriharmonic symbol of exponential type must be individually of the same type, as in the case of
general pluriharmonic symbols (see Section 2.5). So, according to the equivalence in Proposition 2.7, we make the following assumption throughout the discussion unless otherwise specified:

\[(FB) \quad B[f \bar{g} + h \bar{k}] = f \bar{g} + h \bar{k}\]

for \(f, g, h, k \in Hol(\mathbb{C}) \cap E(\mathbb{C})\).

Put \(u := f \bar{g} + h \bar{k}\). Since functions under consideration are all exponential type, Proposition 3.9 gives a representation of the form

\[
\sum_{j=0}^{\infty} \frac{(\lambda \Delta)^j}{j!} u(z) = \sum_{m=-L}^{L} \varphi_m(z)e^{2\pi im\lambda}, \quad \lambda, z \in \mathbb{C}
\]

for some integer \(L \geq 0\). Given an integer \(j \geq 1\), comparing coefficients of \(\lambda^j\) of both sides, we obtain a system of equations

\[
\Delta^j u(z) = \sum_{m=-L}^{L} \varphi_m(z)(2\pi im)^j, \quad j = 0, 1, \ldots, 2L + 1,
\]

or more explicitly in a matrix form,

\[
\begin{pmatrix}
  u(z) \\
  \Delta u(z) \\
  \vdots \\
  \Delta^{2L+1} u(z)
\end{pmatrix} = \begin{pmatrix}
  1 & \cdots & 1 \\
  -2\pi i L & \cdots & 2\pi i L \\
  \vdots & \vdots & \vdots \\
  (-2\pi i L)^{2L+1} & \cdots & (2\pi i L)^{2L+1}
\end{pmatrix} \begin{pmatrix}
  \varphi_{-L}(z) \\
  \varphi_{-L+1}(z) \\
  \vdots \\
  \varphi_L(z)
\end{pmatrix}.
\]

Since the rows of the matrix on the right hand side are linearly dependent, one concludes that there is a non-trivial relation of the form

\[
\sum_{\ell=0}^{2L+1} \gamma_{\ell} \Delta^\ell u(z) = 0
\]

for some coefficients \(\gamma_{\ell}\), not all 0. This can be rephrased as a more explicit equation of the form

\[
\sum_{j=1}^{J} \gamma_{n_j} \left[ f^{(n_j)}(z)g^{(n_j)}(z) + h^{(n_j)}(z)k^{(n_j)}(z) \right] = 0
\]

where coefficients \(\gamma_{n_j}\) are all nonzero. Now, using Lemma 2.9, we obtain an equation

\[(4.1) \quad \sum_{j=1}^{J} \gamma_{n_j} \left[ f^{(n_j)}(z)(g^*)^{(n_j)}(w) + h^{(n_j)}(z)(k^*)^{(n_j)}(w) \right] = 0\]

in the independent variables \(z, w \in \mathbb{C}\). We may assume (when \(J \geq 2\)) that \(n_j < n_{j+1}\) for all \(j\). To simplify notation we also express (4.1) in a vector form. Put

\[
V := \begin{pmatrix}
  \partial^{n_1} \\
  \vdots \\
  \partial^{n_J}
\end{pmatrix}_{J \times 1} \quad \text{and} \quad W := \begin{pmatrix}
  \gamma_{n_1} \partial^{n_1} \\
  \vdots \\
  \gamma_{n_J} \partial^{n_J}
\end{pmatrix}_{J \times 1}.
\]
Also, put \( V_f := Vf \) and \( W_f := Wf \), etc. Then, (4.1) becomes

\[(4.2) \quad W^t_{g^*}(w)V_f(z) + W^t_k(w)V_h(z) = 0,\]

where the super-script \( t \) means the transpose.

We introduce further notation. Recall that \( \mathcal{A}_1 \) denotes the algebra generated by the polynomial algebra \( \mathcal{P}_1 \) and the reproducing kernel algebra \( \mathcal{K}_1 \); see (3.3). Put

\[ D_1 := \{ p(\partial) : 0 \neq p \in \mathcal{P}_1 \} \]

for the collection of all linear differential operators with constant coefficients. From elementary theory of the ordinary differential equations with constant coefficients, note

\[ \mathcal{A}_1 = \bigcup_{D \in \mathcal{D}_1} \ker D \]

where kernels are taken in \( \mathcal{H}o(\mathbb{C}) \).

We now pause to characterize semi-commuting Toeplitz operators with symbols under consideration. In the next theorem the possibility of the second case of (c) is an extra case for the Fock space, which has no analogue on the Bergman space over the ball or the polydisk; see [10, 26].

**Theorem 4.1.** Given \( f, g \in \mathcal{H}o(\mathbb{C}) \cap \mathcal{E}(\mathbb{C}) \), the following statements are equivalent:

(a) \( [T_f, T_g] = 0 \).
(b) \( \mathcal{B}[f \overline{g}] = f \overline{g} \).
(c) Either (i) or (ii) holds;
   (i) \( f \) or \( g \) is constant.
   (ii) There are finite collections \( \{a_j\}_{j=1}^N \) and \( \{b_\ell\}_{\ell=1}^M \) of distinct complex numbers such that

\[ f \in \text{span}\{K_{a_1}, \ldots, K_{a_N}\} \quad \text{and} \quad g \in \text{span}\{K_{b_1}, \ldots, K_{b_M}\} \]

with \( a_j b_\ell \in 2\pi i \mathbb{Z} \) for each \( j \) and \( \ell \).

**Proof.** By Proposition 2.7 and Theorem 3.5 we only need to prove the implication (b) \( \implies \) (c). So, assume (b). If \( f \) or \( g \) is a polynomial, then Corollary 3.12 implies that (i) holds. So, assume that neither \( f \) nor \( g \) is a polynomial. By (4.2) (with \( h = 0 \) or \( k = 0 \)) we have

\[ W^t_{g^*}(w)V_f(z) = 0, \]

which is the same as

\[ [W^t_{g^*}(w)V] f(z) = 0 \quad \text{or} \quad [W^t_f(z)V] g^*(w) = 0. \]

So, when \( w \) is fixed this is a linear differential equation in \( z \) with constant coefficients, and vice versa. Also, note \( J \geq 2 \), because \( f \) and \( g^* \) are not polynomials. Thus, fixing \( w \) such that \( W^t_{g^*}(w) \neq 0 \), we see that \( f \) must be of the form

\[ f = \sum_{j=1}^N p_j K_{a_j} \in \mathcal{A}_1 \setminus \mathcal{P}_1, \]

where \( p_j \in \mathcal{P}_1 \) for each \( j \) and \( a_j \)'s are distinct complex numbers. Exchanging the roles of \( z \) and \( w \), we see that \( g \) also must be of the same form, i.e.,

\[ g = \sum_{\ell=1}^M q_\ell K_{b_\ell} \in \mathcal{A}_1 \setminus \mathcal{P}_1, \]
where \( q_\ell \in \mathcal{P}_1 \) for each \( \ell \) and \( b_\ell \)'s are distinct complex numbers. We see from Theorem 3.5 that (ii) holds. Overall, we conclude that (b) implies (c). The proof is complete. \( \square \)

As an application we obtain a zero-product property as in the next theorem. We do not know whether the hypothesis \( T_uT_v = 0 = T_vT_u \) can be relaxed to \( T_uT_v = 0 \).

**Theorem 4.2.** For \( f, g, h, k \in Hol(\mathbb{C}) \cap E(\mathbb{C}) \) put \( u := f + \overline{k} \) and \( v := h + \overline{g} \). If \( T_uT_v = 0 = T_vT_u \), then either \( u = 0 \) or \( v = 0 \).

**Proof.** Assume \( T_uT_v = 0 = T_vT_u \). We claim that at least one of \( f, g, h, k \) is constant. With this claim granted, it is easily seen that either \( u = 0 \) or \( v = 0 \). For example, assume that \( g \) is constant. Then, since \( v \) is holomorphic, we have \( 0 = T_uT_v = T_{uv} \), which implies \( uv = 0 \). So, we conclude by real-analyticity that either \( u = 0 \) or \( v = 0 \). Other cases can be treated similarly.

We now proceed to prove that at least one of \( f, g, h, k \) is constant. To derive a contradiction, assume that none of them is constant. By a straightforward calculation using Lemma 2.4, the assumption \( T_uT_v = 0 = T_vT_u \) is equivalent to

\[
\mathcal{B}[h\overline{k}] = -fh - \overline{gk} - f\overline{g}
\]

and

\[
\mathcal{B}[f\overline{g}] = -fh - \overline{gk} - h\overline{k}.
\]

Applying the Berezin transform to both sides of (4.4), we have \( \mathcal{B}_2[f\overline{g}] = f\overline{g} \) by (4.3), which can be rephrased via an elementary change of variables as \( \mathcal{B}[(f\overline{g})_{\sqrt{2}}] = (f\overline{g})_{\sqrt{2}} \), where \( (f\overline{g})_{\sqrt{2}} \) denotes the dilated function \( z \mapsto (f\overline{g})(\sqrt{2}z) \). It follows from Theorem 4.1 that there are finite collections of \( \{a_j\}_{j=1}^{N_1} \) and \( \{b_j\}_{j=1}^{M_1} \) of distinct complex numbers (with \( a_j\overline{b}_j \in \pi i\mathbb{Z} \) for all \( j, \ell \)) such that

\[
(4.5) \quad f = \sum_{j=1}^{N_1} \alpha_j K_{a_j} \quad \text{and} \quad g = \sum_{j=1}^{M_1} \beta_j K_{b_j}
\]

for some nonzero coefficients \( \{\alpha_j\} \) and \( \{\beta_j\} \). Similarly, there are finite collections of \( \{c_j\}_{j=1}^{N_2} \) and \( \{d_j\}_{j=1}^{M_2} \) of distinct complex numbers (with \( c_j\overline{d}_j \in \pi i\mathbb{Z} \) for all \( j, \ell \)) such that

\[
(4.6) \quad h = \sum_{j=1}^{N_2} \gamma_j K_{c_j} \quad \text{and} \quad k = \sum_{j=1}^{M_2} \delta_j K_{d_j}
\]

for some nonzero coefficients \( \{\gamma_j\} \) and \( \{\delta_j\} \). We may assume without loss of generality that the numbers \( b_j \) and \( d_j \) are all nonzero.

Using the representations in (4.5), we have \( f\overline{g} = \sum_{j, \ell} \alpha_j \overline{\beta}_\ell \overline{K}_{a_j,b_\ell} \) and no term in this expansion is holomorphic, because \( b_j \neq 0 \) for all \( j \). Also, using the representations in (4.6), we have by Lemma 3.3

\[
(4.7) \quad \mathcal{B}[h\overline{k}] = \sum_{j, \ell} \gamma_j \overline{\delta}_\ell \mathcal{B}[K_{c_j,d_\ell}] = \sum_{j, \ell} \gamma_j \overline{\delta}_\ell e^{c_j\overline{d}_\ell} K_{c_j,d_\ell}.
\]

Note again that no term in the above sum is holomorphic, because \( d_j \neq 0 \) for all \( j \). Thus, comparing the holomorphic parts of both sides of (4.3), we see that \( fh = (\text{constant}) \) and this
constant must be nonzero, because \( f \) and \( h \) are not identically zero. Now, since \( f, h \in Hol(C) \)
are nonvanishing and of exponential type, we have
\[
f(z) = f_0e^{az} \quad \text{and} \quad h(z) = h_0e^{-az}
\]
for some nonzero numbers \( a, f_0 \) and \( h_0 \). A similar argument yields
\[
g(z) = g_0e^{bz} \quad \text{and} \quad k(z) = k_0e^{-bz}
\]
for some nonzero numbers \( b, g_0 \) and \( k_0 \). Now, we have by (4.7) and (4.3)
\[
h_0k_0e^{a-b}K_{-a,-b} + f_0h_0 + g_0k_0 = 0,
\]
which is a contradiction to the fact that \( \{K_{a,b}, K_{-a,-b}, 1\} \) is linearly independent. This
completes the proof. \( \square \)

**Example 4.3.** In the setting of Toeplitz operators \( T_{u_1}, T_{u_2}, T_{u_3} \) with bounded symbols acting
on the Bergman space over the unit disc \( \mathbb{D} \), none of them being a constant multiple of the
identity, Louhichi and Rao conjectured the following in [21]:

(C) If \( [T_{u_1}, T_{u_2}] = 0 = [T_{u_1}, T_{u_3}] \), then \( [T_{u_2}, T_{u_3}] = 0 \).

To our knowledge this conjecture is still open. One may also consider the analogues on
various other settings. In this context, Vasilevski [25] gave a counterexample in the case
of the Bergman space over the multi-dimensional unit ball. Also, Bauer and Issa [5] found
a counterexample, involving at least one unbounded symbol, to the case of the Fock space
over \( C \). Here we wish to point out the failure of (C) in the case of the Fock space over
\( C \), even when one additionally assumes harmonicity of the symbols. For example, consider
nonconstant (co-)holomorphic symbols
\[
u_1(z) := e^z, \quad u_2(z) := e^{\pi z} \quad \text{and} \quad u_3(z) := e^{2\pi i z}.
\]
Clearly, one has \( [T_{u_1}, T_{u_2}] = 0 \), since \( u_1 \) and \( u_2 \) are holomorphic. Moreover, Theorem 4.1
implies that \( [T_{u_1}, T_{u_3}] = -(T_{u_1}, T_{u_3}) = 0 \). On the other hand, Theorem 4.1 again implies that
\( [T_{u_2}, T_{u_3}] = -(T_{u_2}, T_{u_3}) \neq 0 \), since \( 2\pi i \notin 2\pi i \mathbb{Z} \).

**Remark 4.4.** Note that the implication (b) \( \Rightarrow \) (c) of Theorem 4.1 does not generalize to
higher dimensions \( n > 1 \). In fact, harmonic functions of the form \( f\overline{g} \) with \( f, g \in Hol(C^n) \cap
\mathcal{E}(C^n) \) are all fixed by the Berezin transform.

We now continue to investigate properties of solutions of (FB). We consider the following
three cases separately:

(A) All of \( f, g, h, k \) belong to \( \mathscr{A}_1 \).

(S) Some, but not all, of \( f, g, h, k \) belong to \( \mathscr{A}_1 \).

(N) None of \( f, g, h, k \) belongs to \( \mathscr{A}_1 \).

First, we consider solutions as in (A) of (FB). For harmonic functions as in property (ii) of
the next proposition, recall that there are more explicit characterizations; see Remark 3.13.

**Proposition 4.5.** Assume (A). Then (FB) holds if and only if one of the following two
cases is fulfilled:

(i) \( B[f\overline{g}] = f\overline{g} \) and \( B[h\overline{k}] = h\overline{k} \).

(ii) There are functions \( A, B \in \mathscr{A}_1 \) such that \( B[AB] = AB \) and \( f\overline{g} + h\overline{k} - AB \) is harmonic.
Proof. Before proceeding, we first introduce some temporary notation. We denote by \( E_f \) the non-polynomial part of \( f \) so that
\[
P_f := f - E_f \in \mathcal{P}_1
\]
is the polynomial part of \( f \). We define \( E_g, P_g, \) etc, in a similar way.

Since the Berezin transform fixes harmonic functions of exponential type, the sufficiency is clear. We now prove the necessity. Assume (FB). Using the polynomial and the non-polynomial parts, decompose
\[
f\overline{g} + h\overline{k} = (P_f \overline{P}_g + P_h \overline{P}_k) + (P_f \overline{E}_g + P_h \overline{E}_k) + (E_f \overline{P}_g + E_h \overline{P}_k) + (E_f \overline{E}_g + E_h \overline{E}_k).
\]
By Theorem 3.5 the Berezin transform of each group is invariant. Namely, we have
\[
\begin{align*}
(PP) \quad &\mathcal{B}[P_f \overline{P}_g + P_h \overline{P}_k] = P_f \overline{P}_g + P_h \overline{P}_k, \\
(PE) \quad &\mathcal{B}[P_f \overline{E}_g + P_h \overline{E}_k] = P_f \overline{E}_g + P_h \overline{E}_k, \\
(EP) \quad &\mathcal{B}[E_f \overline{P}_g + E_h \overline{P}_k] = E_f \overline{P}_g + E_h \overline{P}_k, \\
(EE) \quad &\mathcal{B}[E_f \overline{E}_g + E_h \overline{E}_k] = E_f \overline{E}_g + E_h \overline{E}_k. 
\end{align*}
\]
We split the proof into four cases.

(CASE 1): Suppose that both pairs \( \{f, g\} \) and \( \{h, k\} \) contain a polynomial. In this case we see from Corollary 3.12 that (ii) holds.

(CASE 2): Suppose that both pairs \( \{E_f, E_h\} \) and \( \{E_g, E_k\} \) are linearly independent. Then, from (EE) together with Theorem 3.5, we have
\[
\mathcal{B}[E_f \overline{E}_g] = E_f \overline{E}_g \quad \text{and} \quad \mathcal{B}[E_h \overline{E}_k] = E_h \overline{E}_k.
\]
Also, from (PE) and (EP), we see that \( P_f, P_h, P_g, P_k \) are all constants again by Theorem 3.5. Thus (i) holds.

(CASE 3): Suppose that only one of the pairs \( \{E_f, E_h\} \) and \( \{E_g, E_k\} \) is linearly independent. Without loss of generality assume that \( \{E_f, E_h\} \) is linearly independent but \( \{E_g, E_k\} \) is linearly dependent. Note from (CASE 1) that we may assume either \( E_g \neq 0 \) or \( E_k \neq 0 \). We may thus further assume \( E_g \neq 0 \) and \( E_k = cE_g \) for some constant \( c \). Since \( \{E_f, E_h\} \) is linearly independent, we see from (EP) that \( P_g \) and \( P_k \) are constants, as in the proof of (CASE 2). Thus \( k - cg = c_2 \) where \( c_2 = P_k - cP_g \). Also, it follows from (PE) that \( P_f + cP_h = c_1 \) for some constant \( c_1 \) and thus \( f + c_1h = E + c_1 \) where \( E := E_f + cE_h \). Accordingly, we have
\[
f\overline{g} + h\overline{k} = E\overline{g} + c_1\overline{g} + c_2\overline{h}.
\]
Note \( \mathcal{B}[E\overline{E}_g] = E\overline{E}_g \) by (EE) and hence \( \mathcal{B}[E\overline{g}] = E\overline{g} \). So, (ii) holds in this case.

(CASE 4): Suppose that both pairs \( \{E_f, E_h\} \) and \( \{E_g, E_k\} \) are linearly dependent. We may assume by (CASE 1) that both \( E_f \) and \( E_g \) are nontrivial. Thus, by the linear dependence, we have \( E_h = cE_f \) and \( E_k = dE_g \) for some constants \( c \) and \( d \). It follows from (PE) and (EP) that
\[
P_f + dP_h = c_1 \quad \text{and} \quad P_g + cP_k = c_2
\]
for some constants \( c_1 \) and \( c_2 \). This, together with (PP), yields
\[
\mathcal{B}[(1 + cd)P_h \overline{P}_k] = (1 + cd)P_h \overline{P}_k.
\]
First, assume \( cd + 1 = 0 \). We then have
\[
E_f = \frac{1}{c} E_h = -\bar{d} E_h \quad \text{and} \quad E_g = \frac{1}{d} E_k = -\bar{c} E_k
\]
and thus by (4.8)
\[
f + \bar{d} h = c_1 \quad \text{and} \quad g - \frac{1}{d} k = g + \bar{c} k = c_2.
\]
Accordingly, \( f\bar{g} + h\bar{k} \) is harmonic by (3.20) and thus (ii) holds. Next, assume \( cd + 1 \neq 0 \). Then we have \( B [P_h \bar{P}_k] = P_h \bar{P}_k \) by (4.9) and hence, according to Theorem 3.2, either \( P_h \) or \( P_k \) is constant. Assume that \( P_h \) is constant. Then \( P_f \) is also constant by (PE) so that \( h = cf + c_3 \) for some constant \( c_3 \). It follows that
\[
f\bar{g} + h\bar{k} = f(\bar{g} + c\bar{k}) + c_3\bar{k}
\]
and thus (ii) is fulfilled. If \( P_k \) is constant, we can argue in a similar way. This completes the proof.

Next, we consider solutions as in (S) of (FB). We need some preliminary lemmas.

**Lemma 4.6.** Assume (FB). If \( f \in \mathcal{A}_1 \), then \( h \in \mathcal{A}_1 \) or \( k \in \mathcal{P}_1 \).

**Proof.** Assuming \( f \in \mathcal{A}_1 \), we can find some \( D \in \mathcal{D}_1 \) for which \( Df = 0 \). We apply \( D = D_z \) to both sides of (4.2) to get
\[
0 = D_z [W_{k^*}^t (w) V_h(z)] = [W_{k^*}^t (w) V] Dh(z).
\]
Now, either \( k \) is a polynomial or \( W_{k^*} \neq 0 \). In the latter case, fixing a suitable \( w \), we have \( Dh = 0 \) where \( \bar{D} := [W_{k^*}^t (w) V] D \in \mathcal{D}_1 \). Thus \( h \in \mathcal{A}_1 \), as required.

**Lemma 4.7.** Assume (FB) and (S). If \( f, h \in \mathcal{A}_1 \setminus \mathcal{P}_1 \), then \( g, k \notin \mathcal{A}_1 \) and \( cf + dh \in \mathcal{P}_1 \) for some constant \( c \) and \( d \), not both 0.

**Proof.** Pick \( D \in \mathcal{D}_1 \) of minimal order such that \( f, h \in \ker D \). By factorization (up to a constant factor) we can rewrite \( D \) in the form
\[
D = \prod_{j=1}^M (\partial - \lambda_j)^{m_j},
\]
where \( \lambda_1, \ldots, \lambda_M \) are distinct complex numbers and \( m_j \in \mathbb{N} \) for each \( j \). Since \( f \) and \( h \) are not polynomials by assumption, there is some \( j_0 \) with \( \lambda_{j_0} \neq 0 \). Without any restriction we assume \( j_0 = 1 \) and consider the differential operator
\[
\hat{D} := (\partial - \lambda_1)^{m_1-1} \prod_{j=2}^M (\partial - \lambda_j)^{m_j}.
\]
Also, regarding \( D \) as an operator acting on \( \mathcal{A}_1 \), fix a basis \( \{ p_{j,\ell} \in \mathcal{A}_1 : j = 1, \ldots, M \text{ and } \ell = 1, \ldots, m_j \} \) of \( \ker D \) such that
\[
\text{span} \{ p_{j,\ell} \} = \ker (\partial - \lambda_j) \ell \ominus \ker (\partial - \lambda_j) \ell-1
\]
for each \( j = 1, \ldots, M \) and \( \ell = 1, \ldots, m_j \). For short we rewrite such basis as \( \{ q_1, \ldots, q_N \} \) with \( q_1 = p_{1, m_1} \). Since \( f, g \in \ker D \), we can form the expansions
\[
f = \sum_{\ell=1}^N c_\ell q_\ell \quad \text{and} \quad h = \sum_{\ell=1}^N d_\ell q_\ell.
\]
for some coefficients $c_\ell$ and $d_\ell$. Note either $c_1 \neq 0$ or $d_1 \neq 0$ by minimality of $D$. Inserting these into (4.2), we obtain

$$0 = \sum_{\ell=1}^{N} c_\ell W^t_{g^*}(w)V_{q_\ell}(z) + \sum_{\ell=1}^{N} d_\ell W^t_{k^*}(w)V_{q_\ell}(z)$$

$$= \sum_{\ell=1}^{N} W^t_{c_\ell g^* + d_\ell k^*}(w)V_{q_\ell}(z).$$

Note from the construction that $q_2, \ldots, q_N$ are all annihilated by $\tilde{D}$. Now, an application of $D = \tilde{D}$ to both sides yields

$$0 = W^t_{c_1 g^* + d_1 k^*}(w)V_{\tilde{D}q_1}(z) = \left[ W^t_{\tilde{D}q_1}(z)V \right](c_1 g^* + d_1 k^*)(w).$$

Note $W_{\tilde{D}q_1}(z)$ is never zero, because $\tilde{D}q_1 \in \ker(\partial - \lambda_1)$ is a constant multiple of $e^{\lambda_1 z}$ with $\lambda_1 \neq 0$. Thus (4.10) shows that $c_1 g^* + d_1 k^* \in \mathcal{A}_1$. Since either $c_1 \neq 0$ or $d_1 \neq 0$, we may assume $c_1 = 1$ for simplicity so that $g^* + d_1 k^* \in \mathcal{A}_1$, or said differently, $g + d_1 k \in \mathcal{A}_1$. Thus $g, k \notin \mathcal{A}_1$ by (S). Note

$$\bar{f}g + \bar{h}k = (g + \bar{d}_1 k)\bar{f} + k(\bar{h} - \bar{d}_1 \bar{f}).$$

Thus the right hand side is also fixed by the Berezin transform. Since $g + \bar{d}_1 k \in \mathcal{A}_1$ and $k \notin \mathcal{A}_1$, we deduce from Lemma 4.6 that $h - d_1 f \in \mathcal{P}_1$. This concludes the proof. \hfill \Box

**Lemma 4.8.** Assume (FB) and (S) with $f, g, h, k$ all nonconstant. Assume that at least one of $f, g, h, k$ is a polynomial. Then $f\bar{g} + h\bar{k}$ is harmonic.

**Proof.** Assume that $f, g, h, k$ are all nonconstant and, without loss of generality, assume $h \in \mathcal{P}_1$. We will show that either $f$ or $g$ is a polynomial and thus conclude the lemma by Corollary 3.12.

Suppose that neither $f$ nor $g$ is a polynomial. We will show that $f, g, h, k \in \mathcal{A}_1$, which is impossible by (S). An application of $\partial^N$ with $N := \deg h$ to both sides of (4.2) yields

$$[W^t_g(w)V]\partial^{N+1}f(z) = 0$$

for $z, w \in \mathbb{C}$. Note that $W_{g^*}(w)$ is not identically zero, because $g$ is not a polynomial. Thus the differential equation above implies $f \in \mathcal{A}_1$. Similarly, we have $g \in \mathcal{A}_1$. We now proceed to prove $k \in \mathcal{A}_1$. Since $h \in \mathcal{P}_1$, a manipulation as in the proof of Lemma 3.3 yields

$$\mathcal{B}[k\overline{h}](z) = h^*(\partial_z + \bar{z})k(z).$$

Put

$$Q(z) := \mathcal{B}[k\overline{h}](z) - k(z)\overline{h}(z) = [h^*(\partial_z + \bar{z}) - h^*(\bar{z})]k(z)$$

and define

$$(4.11) \quad \tilde{Q}(z, w) := [h^*(\partial_z + w) - h^*(w)]k(z)$$

so that $\tilde{Q}$ is an entire function on $\mathbb{C}^2$ with the property $\tilde{Q}(z, \bar{z}) = Q(z)$. Meanwhile, note from (FB)

$$Q = g\bar{f} - \mathcal{B}[g\bar{f}].$$
Recall \( f, g \in \mathcal{A}_1 \). Thus, using the notation in (3.7), we see from Lemma 3.3 that \( Q \) is of the form
\[
Q(z) = \sum_{j, \ell \text{ finite}} \left[ p_j(z)q_{\ell}(z) - R_{a_j, b_{\ell}, p_j, q_{\ell}}(z, \bar{z}) \right] K_{a_j, b_{\ell}}(z)
\]
with holomorphic polynomials \( p_j, q_{\ell} \) and distinct \((a_j, b_{\ell}) \in \mathbb{C}^2\). This representation of \( Q \), together with Lemma 2.9, yields
\[
\tilde{Q}(z, w) = \sum_{j, \ell \text{ finite}} \left[ p_j(z)q_{\ell}^*(w) - R_{a_j, b_{\ell}, p_j, q_{\ell}}(z, w) \right] K_{a_j}(z)K_{b_{\ell}}^*(w).
\]
Accordingly, we see that \( \tilde{Q}(., w) \in \mathcal{A}_1 \) for each fixed \( w \in \mathbb{C} \). Thus, for each fixed \( w \in \mathbb{C} \), we can choose a differential operator \( D(w) \in \mathcal{D} \) which annihilates \( \tilde{Q}(., w) \). Accordingly, we obtain by (4.11)
\[
D(w) \left[ h^*(\partial_z + w) - h^*(w) \right] k = D(w)\tilde{Q}(., w) = 0.
\]
Recall that \( h \) is nonconstant by assumption. So, choosing a suitable \( w \) for which \( h^*(\partial_z + w) \neq h^*(w) \), we conclude \( k \in \mathcal{A}_1 \), as asserted. This completes the proof.

Assuming \((S)\) and combining the results in Lemmas 4.6-4.8, we obtain the following properties of solutions of \((FB)\).

**Proposition 4.9.** Assume \((S)\) with \( f, g, h, k \) all nonconstant. Then \((FB)\) holds if and only if there are functions \( A, B \in \text{Hol}(\mathbb{C}) \cap \mathcal{E}(\mathbb{C}) \) such that \( \mathcal{B}[AB] = AB \) and \( f\overline{g} + h\overline{k} - \overline{AB} \) is harmonic.

**Proof.** The sufficiency is clear as in the proof of Proposition 4.5. We now prove the necessity. Assume \((FB)\). By Lemma 4.8 we may assume none of \( f, g, h, k \) is a polynomial. We may also assume \( f \in \mathcal{A}_1 \) without loss of generality. Since \( f \in \mathcal{A}_1 \) and \( k \notin \mathcal{P}_1 \), we have \( h \in \mathcal{A}_1 \) by Lemma 4.6. According to Lemma 4.7, we have
\[
g, k \notin \mathcal{A}_1 \quad \text{and} \quad p := cf + dh \in \mathcal{P}_1
\]
for some constants \( c \) and \( d \), not both 0. We may assume that \( d = 1 \) so that
\[
f\overline{g} + h\overline{k} = f(\overline{g} - c\overline{k}) + p\overline{k}.
\]
So, if \( p \) or \( g - c\overline{k} \) is constant, then the conclusion clearly holds. Otherwise, applying Lemma 4.8 to the right hand side of the above, we deduce that \( f\overline{g} + h\overline{k} \) is harmonic. This completes the proof.

Finally, we consider solutions as in \((N)\) of \((FB)\).

**Proposition 4.10.** Assume \((N)\). Then \((FB)\) holds if and only if \( f\overline{g} + h\overline{k} \) is harmonic.

**Proof.** The sufficiency being again clear, we only need to prove the necessity. Assume \((FB)\). We claim
\[
p := h - cf \in \mathcal{P}_1 \quad \text{and} \quad q := g - dk \in \mathcal{P}_1
\]
for some (nonzero) constant \( c \) and \( d \). With this granted, we can write
\[
f\overline{g} + h\overline{k} = f(\overline{dk + q + c\overline{k}}) + p\overline{k}.
\]
If \( p \) is constant, then we see from \((FB)\) that \( f(\overline{dk + q + c\overline{k}}) \) is fixed by the Berezin transform and thus \( f \in \mathcal{A}_1 \) by Theorem 4.1. So, \( p \) cannot be constant by \((N)\). Thus we can apply Lemma 4.8 to the right hand side of the above to conclude the proposition.
It remains to prove (4.12). Note that \( \{f^{(n_1)}, \ldots, f^{(n_J)}\} \) is linearly independent as a set of functions, because \( f \notin \mathcal{A}_1 \) by (N). This equivalently means that \( \mathbb{C}^J \oplus Y_f = \{0\} \) where

\[
Y_f := \text{span} \{V_f(z) \in \mathbb{C}^J : z \in \mathbb{C}\}.
\]

In other words, \( \dim Y_f = J \). Thus, choosing \( \lambda_1, \ldots, \lambda_J \in \mathbb{C} \) such that \( V_f(\lambda_1), \ldots, V_f(\lambda_J) \) form a basis of \( Y_f \), we see that the matrix

\[
A_f := [V_f(\lambda_1), \ldots, V_f(\lambda_J)]_{J \times J}
\]

is invertible. Define (possibly non-invertible) \( A_h \) by replacing \( f \) with \( h \) in (4.13). Then put \( \Phi := (A_f)^{-1}V_f \in \text{Hol}(\mathbb{C}^J) \) so that

\[
V_f(z) = A_f \Phi(z).
\]

Applying (4.2) to each \( z = \lambda_j \) and then rewriting all \( J \) equations as a single matrix equation, we obtain

\[
W_{g^*}(w)A_f + W_{k^*}(w)A_h = 0.
\]

From (4.14) and (4.2) we thus have

\[
0 = W_{g^*}(w)V_f(z) - W_{g^*}(w)A_f \Phi(z)
= -W_{k^*}(w)V_h(z) + W_{k^*}(w)A_h \Phi(z)
= W_{k^*}(w)(A_h \Phi(z) - V_h(z)).
\]

This shows that

\[
A_h \Phi(z) - V_h(z) \in \mathbb{C}^J \oplus \text{span} \{W_{k^*}(w) \in \mathbb{C}^J : w \in \mathbb{C}\} = \{0\};
\]

the last equality holds as above, because \( k \notin \mathcal{A}_1 \) by (N). Since \( z \) is arbitrary, this yields

\[
V_h = A_h \Phi = A_h (A_f)^{-1}V_f.
\]

Accordingly,

\[
k^{(n_j)} \in \text{span} \{f^{(n_1)}, \ldots, f^{(n_J)}\} := S_1
\]

for each \( j = 1, \ldots, J \). Applying \( \partial^{n_J-n_1} \) to the case \( j = 1 \), we see that

\[
k^{(n_j)} \in \text{span} \{f^{(n_j)}, \ldots, f^{(2n_J-n_1)}\} := S_2.
\]

Note that \( \{f^{(n_1)}, \ldots, f^{(n_J)}, \ldots, f^{(2n_J-n_1)}\} \) is also linearly independent by (N) as before. So, we deduce from (4.15) with \( j = J \) and (4.16)

\[
k^{(n_j)} \in S_1 \cap S_2 = \text{span} \{f^{(n_j)}\},
\]

which implies the first half of (4.12). Since \( g, h \notin \mathcal{A}_1 \) by (N), one shows the second half of (4.12) in the same way. This completes the proof. \( \square \)

Putting together what we have proved so far, we obtain the following characterization for the solutions of (FB).

**Corollary 4.11.** Let \( f, g, h, k \in \text{Hol}(\mathbb{C}) \cap \mathcal{E}(\mathbb{C}) \). Then (FB) holds if and only if one of the following two cases is fulfilled:

(i) \( \mathcal{B}[fg] = fg \) and \( \mathcal{B}[hk] = hk \).

(ii) There are functions \( A, B \in \text{Hol}(\mathbb{C}) \cap \mathcal{E}(\mathbb{C}) \) such that \( \mathcal{B}[AB] = AB \) and \( fg + hk - AB \) is harmonic.
Proof. The sufficiency is clear as before. For the necessity, assume (FB). If one of $f, g, h, k$ is constant, then (i) holds. Otherwise, either (i) or (ii) holds by Propositions 4.5, 4.9 and 4.10.

In fact Condition (ii) above can be made more explicit via the next lemma.

**Lemma 4.12.** Let $f, g, h, k \in \text{Hol}(\mathbb{C}) \cap E(\mathbb{C})$ and assume that none of them is constant. Assume that there are functions $A, B \in \text{Hol}(\mathbb{C}) \cap E(\mathbb{C})$ such that $B[A\overline{B}] = A\overline{B}$ and $f\overline{g} + h\overline{k} - A\overline{B}$ is harmonic. Then one of the following three cases is fulfilled:

(i) $f\overline{g} + h\overline{k}$ is harmonic.

(ii) There are constants $a, b$ and $c$ with $ab \neq 0$ such that

$$af - bh = c \quad \text{and} \quad B[f(b\overline{g} + a\overline{k})] = f(b\overline{g} + a\overline{k}).$$

(iii) There are constants $a, b$ and $c$ with $ab \neq 0$ such that

$$ag - bk = c \quad \text{and} \quad B[(\overline{bf} + \overline{ah})\overline{g}] = (\overline{bf} + \overline{ah})\overline{g}.$$

Proof. If $A$ or $B$ is constant, then (i) holds. So, assume that both $A$ and $B$ are nonconstant.

By harmonicity and Lemma 2.9 we have

$$f'(z)\overline{g'}(w) + h'(z)\overline{k'}(w) = A'(z)\overline{B'(w)}$$

for any $z, w \in \mathbb{C}$. Put $F := \frac{f'}{A'}, G := \frac{g'}{B'}, H := \frac{k'}{A'}$ and $K := \frac{k'}{B'}$ for simplicity. Since $A'$ and $B'$ are not identically zero, we see from the above

$$F(z)G(\overline{w}) + H(z)K(\overline{w}) = 1. \tag{4.18}$$

From this one can see that functions $F$ and $H$ are both constant or both nonconstant. The same is true for functions $G$ and $K$.

First, consider the case where $F$ and $H$ are nonconstant. Taking $z$-derivative of both sides of (4.18) and then rearranging the resulting equation, we obtain

$$\frac{K(w)}{G(w)} + \frac{F'(z)}{H'(z)} = 0,$$

which implies $K = c_1G$ for some constant $c_1$. Inserting this into (4.18), we obtain $[F(z) + c_1H(z)]G(\overline{w}) = 1$ and hence deduce that $G$ and $K$ must be constant. Now, writing

$$g = bB + b' \quad \text{and} \quad k = aB + a'$$

where $a, b, a'$ and $b'$ are constants with $ab \neq 0$, we see that the first part of (iii) holds. Meanwhile, since $A = \overline{bf} + \overline{ah} + c'$ for some constant $c'$ by (4.17), we have

$$\overline{bA\overline{B}} - (\overline{bf} + \overline{ah})\overline{g} = -\overline{b'(bf + ah)} + c'(\overline{g} - \overline{b'}).$$

In particular, the function on the left hand side of the above is harmonic and thus is fixed by the Berezin transform. Now, since $B[A\overline{B}] = A\overline{B}$ by assumption, we see that the second part of (iii) also holds.

Next, when $G$ and $K$ are nonconstant, the same argument shows that (ii) holds. Finally, when functions $F, G, H$ and $K$ are all constant, the argument above shows that both (ii) and (iii) hold. The proof is complete.

In conjunction with Lemma 4.12, we note the following.
Lemma 4.13. Let \( f, g, h, k \in Hol(\mathbb{C}) \cap E(\mathbb{C}) \). If there are constants \( a, b \) and \( c \) with \( ab \neq 0 \) such that
\[
af - bh = c \quad \text{and} \quad B[f(bg + ak)] = f(bg + ak),
\]
then
\[
B[fg + h\bar{k}] = f\bar{g} + h\bar{k}.
\]

Proof. Suppose (4.19). Note
\[
b(fg + hk) = f(bg + ak) - ck.
\]
Since \( f(bg + ak) \) is fixed by the Berezin transform by assumption and \( c\bar{k} \) is harmonic, we see that the function on the left hand side of the above is fixed by the Berezin transform. Since \( b \neq 0 \), this completes the proof. \( \square \)

We finally obtain the following characterization for the solutions of (FB).

Theorem 4.14. Let \( f, g, h, k \in Hol(\mathbb{C}) \cap E(\mathbb{C}) \). Then (FB) holds if and only if one of the following four cases is fulfilled:

(i) \( B[fg] = fg \) and \( B[h\bar{k}] = h\bar{k} \).

(ii) \( f\bar{g} + h\bar{k} \) is harmonic.

(iii) There are constants \( a, b \) and \( c \) with \( ab \neq 0 \) such that
\[
af - bh = c \quad \text{and} \quad B[f(bg + ak)] = f(bg + ak).
\]

(iv) There are constants \( a, b \) and \( c \) with \( ab \neq 0 \) such that
\[
ag - bk = c \quad \text{and} \quad B[(\bar{b}f + \bar{a}h)\bar{g}] = (\bar{b}f + \bar{a}h)\bar{g}.
\]

Proof. When none of \( f, g, h \) and \( k \) is constant, the theorem is immediate from Corollary 4.11, Lemma 4.12 and Lemma 4.13. Otherwise, (FB) is clearly equivalent to (i). \( \square \)

As a consequence, we have the following characterization of commuting Toeplitz operators with symbols under consideration.

Theorem 4.15. Given \( f, g, h, k \in Hol(\mathbb{C}) \cap E(\mathbb{C}) \), the following three statements are equivalent:

(a) \( [T_{f+\overline{f}}, T_{h+\overline{h}}] = 0 \).

(b) \( B[fg - h\overline{k}] = fg - h\overline{k} \).

(c) One of the following four conditions is fulfilled:

(i) \( (T_f, T_{\overline{f}}) = (T_h, T_{\overline{h}}) = 0 \).

(ii) \( \{f + \overline{f}, h + \overline{h}, 1\} \) is linearly dependent.

(iii) There are constants \( a, b \) and \( c \) with \( ab \neq 0 \) such that
\[
af + bh = c \quad \text{and} \quad B[f(bg + ak)] = f(bg + ak).
\]

(iv) There are constants \( a, b \) and \( c \) with \( ab \neq 0 \) such that
\[
ag + bk = c \quad \text{and} \quad B[(\bar{b}f + \bar{a}h)\bar{g}] = (\bar{b}f + \bar{a}h)\bar{g}.
\]

Proof. The equivalence (a) \( \iff \) (b) holds by Proposition 2.7. The equivalence (b) \( \iff \) (c) holds by Theorem 4.14 together with Theorem 4.1 and (3.20). \( \square \)
As was mentioned earlier before Theorem 4.1, compared to the corresponding result in [2] for the Bergman space over the unit disk, the first condition of (c) above contains extra cases for the Fock space. Note that the last two conditions of (c) above also contain extra cases for the Fock space. For example, using Theorem 4.1, pick any functions \( f, \varphi, \psi \in \text{Hol}(\mathbb{C}) \cap E(\mathbb{C}) \) such that \( B[f \varphi] \neq f \varphi \) and \( B[f \psi] = f \psi \). One may then check \( [T_{f+\varphi+\psi}, T_{f+\varphi-\psi}] = 0 \) by Theorem 4.15.

Acknowledgement: The first-named author acknowledges financial support through the funds NRF(2013R1A1A2004736) and NRF(2012R1A1A2000705) of Korea which enabled him to visit the Korea University, Seoul where the discussion on commuting Toeplitz operators with pluriharmonic symbols on the Fock space has been started.

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