PATH COMPONENTS OF COMPOSITION OPERATORS
OVER THE HALF-PLANE

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Abstract. The characterization of path components in the space of composition operators acting in various settings has been a long-standing open problem. Recently Dai has obtained a characterization of when two composition operators acting on the weighted Hilbert-Bergman space on the unit disk are linearly connected, i.e., they are joined by a continuous “line segment” of composition operators induced by convex combinations of the maps inducing the two given composition operators.

In this paper we consider composition operators acting on the weighted Bergman spaces over the half-plane. Since not all composition operators are bounded in this setting, we introduce a metric induced by the operator norm and study when (possibly unbounded) composition operators are linearly connected in the resulting metric space. We obtain necessary conditions that under a natural additional assumption are also sufficient. We also study the problem of when a composition operator is isolated. Complete results are obtained for composition operators induced by linear fractional self-maps of the half-plane. We show that the only such composition operators that are isolated are those induced by automorphisms of the half-plane. We also characterize when composition operators induced by linear fractional self-maps belong to the same path component. The characterization demonstrates that composition operators in the same path component may have inducing maps with different behavior at infinity. In contrast, when the setting is the disk the corresponding boundary behavior of the inducing maps must match.

1. Introduction

Let \( H \) be the upper half of the complex plane \( \mathbb{C} \), i.e.,
\[
H := \{ z \in \mathbb{C} : \text{Im} \ z > 0 \}
\]
and let \( \mathcal{S} \) be the class of all holomorphic self-maps of \( H \). Each \( \varphi \in \mathcal{S} \) induces a composition operator \( C_\varphi \) defined by
\[
C_\varphi f := f \circ \varphi
\]
for functions $f$ holomorphic on $H$. It is clear that $C_\varphi$ takes the space of holomorphic functions on $H$ into itself. An extensive study on the theory of composition operators has been established during the past four decades on various settings. We refer to standard references [8] and [18] for various aspects on the theory of composition operators acting on holomorphic function spaces.

For $\alpha > -1$, put

$$dA_\alpha(z) := (\text{Im } z)^\alpha \, dA(z)$$

where $A$ is the area measure on $H$. In this definition, normalizing constants, which are usually inserted for brevity of reproducing kernels, are omitted for convenience.

For $0 < p < \infty$, we denote by $A^p_\alpha(H)$ the weighted Bergman space consisting of all holomorphic functions $f$ on $H$ such that the “norm”

$$\|f\|_{A^p_\alpha} := \left\{ \int_H |f|^p \, dA_\alpha \right\}^{1/p}$$

is finite. As is well known, each space $A^p_\alpha(H)$ is a closed subspace of $L^p(dA_\alpha)$. Thus $A^p_\alpha(H)$ is a Banach space for $1 \leq p < \infty$. Also, when $0 < p < 1$, the space $A^p_\alpha(H)$ is a complete metric space under the translation-invariant metric $(f,g) \mapsto \|f - g\|_{A^p_\alpha}^p$.

One may easily check that the spaces $A^p_\alpha(H)$ are not Möbius invariant. So, it is certain that not all composition operators are bounded on the spaces $A^p_\alpha(H)$, unlike the disk case. In this regard, Elliott and Wynn [10] obtained a characterization for bounded composition operators on $A^p_\alpha(H)$; see Theorem 2.3 below. More interestingly, they showed that no composition operator on $A^p_\alpha(H)$ is compact; see [19] for a more general result in this direction. Nevertheless, it turns out that there are compact differences of composition operators on $A^p_\alpha(H)$. In fact three of the current authors [5] obtained characterizations for bounded/compact differences of composition operators on $A^p_\alpha(H)$ and demonstrated various examples of distinct composition operators with compact difference, including examples when the individual composition operators are not bounded.

In 1981 Berkson [3] first found the isolation phenomenon for composition operators acting on the Hardy space over the unit disk. Berkson’s isolation result was refined later by Shapiro and Sundberg [20], and also by MacCluer [14]. In the course of their study in [20], Shapiro and Sundberg were naturally led to the question of whether two composition operators belong to the same path component if and only if their difference is compact. While this question was originally for the Hardy spaces, it also initiated similar study on various other settings including the weighted Bergman spaces. It was answered negatively on both the Hardy spaces (see [4, 11, 17]) and the weighted Bergman spaces (see [16]); see also [1] and [15] for similar results on different settings. In spite of such negative results, the Shapiro-Sundberg question
initiated the problem of characterizing path components of composition operators, which is now a long-standing open problem. In connection with that problem, Dai [9] has quite recently obtained a complete characterization of when two composition operators acting on the weighted Hilbert-Bergman space over the unit disk are \textit{linearly connected}, i.e., they are joined by a continuous “line segment” of composition operators induced by convex combinations of the maps inducing the two given composition operators. In the current paper we investigate this problem of linear connection in the setting of weighted Bergman spaces over the half-plane. Our work includes linear connection between unbounded composition operators, something that did not come up in Dai’s work since in his setting all composition operators are bounded. Also, our methods apply to general $p$, $0 < p < \infty$, while Dai’s results were restricted to $p = 2$.

To be more explicit, we first introduce some notation. Let $\text{Comp}(\mathbf{H})$ be the class of all composition operators induced by the maps in $\mathcal{S}$. Let $\alpha > -1$ and $0 < p < \infty$. For a linear operator $T$ taking $A^p_\alpha(\mathbf{H})$ into the space of holomorphic functions on $\mathbf{H}$, we put

$$
\|T\|_{A^p_\alpha} := \sup_{\|f\|_{A^p_\alpha} = 1} \|Tf\|_{A^p_\alpha},
$$

which might be possibly $\infty$. When $T$ is bounded on $A^p_\alpha(\mathbf{H})$ with $p \geq 1$, note that $\|T\|_{A^p_\alpha}$ is simply the operator norm on $A^p_\alpha(\mathbf{H})$. For $\varphi, \psi \in \mathcal{S}$, put

$$
d_{\alpha,p}(C_\varphi, C_\psi) := \begin{cases} 
\|C_\varphi - C_\psi\|_{A^p_\alpha} & \text{if } p \geq 1 \\
\|C_\varphi - C_\psi\|^p_{A^p_\alpha} & \text{if } 0 < p < 1
\end{cases}
$$

and define a metric $\widetilde{d}_{\alpha,p}$ on $\text{Comp}(\mathbf{H})$ by

$$
\widetilde{d}_{\alpha,p}(C_\varphi, C_\psi) := \begin{cases} 
d_{\alpha,p}(C_\varphi, C_\psi) & \text{if } \|C_\varphi - C_\psi\|_{A^p_\alpha} < \infty \\
\frac{d_{\alpha,p}(C_\varphi, C_\psi)}{1 + d_{\alpha,p}(C_\varphi, C_\psi)} & \text{if } \|C_\varphi - C_\psi\|_{A^p_\alpha} = \infty
\end{cases}
$$

We denote by $\text{Comp}(A^p_\alpha)$ the metric space $\text{Comp}(\mathbf{H})$ with metric $\widetilde{d}_{\alpha,p}$.

For $\varphi, \psi \in \mathcal{S}$ we refer to the set

$$
[C_\varphi, C_\psi] := \{C_{\varphi_s} : 0 \leq s \leq 1\}
$$

as the \textit{line segment} in $\text{Comp}(\mathbf{H})$ between $C_\varphi$ and $C_\psi$. Following [9], we say that $C_\varphi$ and $C_\psi$ are \textit{linearly connected} in $\text{Comp}(A^p_\alpha)$ if $[C_\varphi, C_\psi]$ is a continuous line segment in $\text{Comp}(A^p_\alpha)$, i.e., if $s \mapsto C_{\varphi_s}$ is a continuous path in $\text{Comp}(A^p_\alpha)$. We also say that $C_\varphi$ and $C_\psi$ are \textit{polygonally connected} in $\text{Comp}(A^p_\alpha)$, if they are connected by a finite number of consecutive continuous line segments in $\text{Comp}(A^p_\alpha)$.

In §2 we collect some basic facts and preliminary results that will be used throughout the paper.

In §3 we study when two composition operators are linearly connected. We first show that if $C_\varphi$ and $C_\psi$ are polygonally connected, then a certain
type of Angular Derivative Cancelation property holds for the pair \( \{ \varphi, \psi \} \) and that \( C_\varphi - C_\psi \) is bounded; see (3.1) and Corollary 3.8. We then show that under an additional hypothesis the converse also holds; see Theorem 3.14. In particular, when \( C_\varphi \) and \( C_\psi \) are both bounded on \( A^p_\alpha(H) \), this results in a characterization of when they are linearly connected or polygonally connected in \( \text{Comp}(A^p_\alpha) \); see Remark 3.15. Another special case is that if \( \varphi \) and \( \psi \) both map to relatively compact subsets of \( H \), then \( C_\varphi \) and \( C_\psi \) are linearly connected in \( \text{Comp}(A^p_\alpha) \) if and only if \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \); see Corollary 3.19.

In §4 we apply the results in §3 to show that the Shapiro-Sundberg question has a negative answer in our setting, and also to study when a composition operator is isolated in \( \text{Comp}(A^p_\alpha) \). We show that \( \text{Im} \varphi \geq \delta > 0 \) is sufficient but not necessary for \( C_\varphi \) to not be isolated; see Corollary 4.5 and Example 4.7. On the other hand, Theorem 4.12 asserts that \( C_\varphi \) is isolated if the set of points in \( \mathbb{R} \) at which \( \varphi \) has a finite angular derivative has positive measure. For linear fractional self-maps of \( H \) we have more complete information. We show in Theorem 4.13 that a composition operator induced by a linear fractional self-map is isolated if and only if the inducing map is an automorphism (=bi-holomorphic onto mapping) of \( H \). Then, in Theorem 4.16, we characterize when composition operators induced by linear fractional self-maps belong to the same path component. The characterization demonstrates that maps with different behavior at infinity may induce composition operators belonging to the same path component. This is quite different from the setting of the disk, where the corresponding boundary behavior of the inducing maps of composition operators in the same path component must agree.

**Constants.** Throughout the paper we use the same letter \( C \) to denote various positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants \( C \) will be often specified in parenthesis. For nonnegative quantities \( X \) and \( Y \) the notation \( X \lesssim Y \) or \( Y \gtrsim X \) means \( X \leq CY \) for some inessential constant \( C \). Similarly, we write \( X \approx Y \) if both \( X \lesssim Y \) and \( Y \lesssim X \) hold.

2. **Prerequisites**

In this section we collect some basic facts and preliminary results to be used throughout the paper.

2.1. **Compact Operator.** It seems better to clarify the notion of compact operators, since the spaces under consideration are not Banach spaces when \( 0 < p < 1 \). Suppose \( X \) and \( Y \) are topological vector spaces whose topologies are induced by complete metrics. A continuous linear operator \( T : X \rightarrow Y \) is said to be compact if the image of every bounded sequence in \( X \) has a subsequence that converges in \( Y \).

We have the following convenient compactness criterion for a linear combination of composition operators acting on the weighted Bergman spaces.
Lemma 2.1. Let $\alpha > -1$ and $0 < p < \infty$. Let $T$ be a linear combination of composition operators and assume that $T$ is bounded on $A^p_\alpha(H)$. Then $T$ is compact on $A^p_\alpha(H)$ if and only if $Tf_n \to 0$ in $A^p_\alpha(H)$ for any bounded sequence $\{f_n\}$ in $A^p_\alpha(H)$ such that $f_n \to 0$ uniformly on compact subsets of $H$.

A proof can be found in [8, Proposition 3.11] for a single composition operator over the unit disk and it can be easily modified for a linear combination over the half-plane.

2.2. Pseudo-hyperbolic Distance. For $z, w \in H$ the pseudohyperbolic distance between $z$ and $w$ is given by

$$\rho(z,w) := \frac{|z - w|}{|z - \overline{w}|}.$$  

Note that $\rho$ is invariant under dilation and horizontal translation. Also, note

$$1 - \rho^2(z,w) = \frac{4(\text{Im } z)(\text{Im } w)}{|z - \overline{w}|^2}.$$  

by a straightforward calculation.

For $z \in H$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic disk centered at $z$ with radius $\delta$. As is well known by the Schwarz-Pick Inequality (see, for example, [13, Theorem 2.1.1]), each holomorphic self-map of $H$ is a $\rho$-contraction. We thus have

$$\varphi(E_\delta(z)) \subset E_\delta(\varphi(z))$$  

for $z \in H$ and $\varphi \in S$.

A straightforward calculation shows that $E_\delta(z)$ is the Euclidean disk with

$$\text{(center)} = x + i \frac{1 + \delta^2}{1 - \delta^2} y \quad \text{and} \quad \text{(radius)} = \frac{2\delta}{1 - \delta^2} y$$  

where $x := \text{Re } z$ and $y := \text{Im } z$. Thus it is easily seen that

$$\frac{1 - \delta}{1 + \delta} < \frac{\text{Im } z}{\text{Im } w} < \frac{1 + \delta}{1 - \delta}$$  

whenever $w \in E_\delta(z)$.

Given $0 < \delta < 1$ and $\alpha > -1$, note that there is a constant $C = C(\alpha, \delta) > 0$ such that

$$C^{-1}(\text{Im } z)^{\alpha+2} \leq A_\alpha[E_\delta(z)] \leq C(\text{Im } z)^{\alpha+2}$$  

for all $z \in H$. This yields the submean value type inequality

$$|f(z)|^p \leq \frac{C}{(\text{Im } z)^{\alpha+2}} \int_{E_\delta(z)} |f|^p dA_\alpha, \quad z \in H$$  

for $0 < p < \infty$ and functions $f$ holomorphic on $H$; see [7, Lemma 3.6] or, for details on the disk, [21, Proposition 4.13].
2.3. Carleson Measure. Let $\mu$ be a locally finite positive Borel measure on $H$. Here, the term *locally finite* means that $\mu(K) < \infty$ for any compact set $K \subset H$. Let $\alpha > -1$ and $0 < \delta < 1$. We put

$$\hat{\mu}^{\alpha,\delta}(z) := \frac{\mu[E_\delta(z)]}{A_\alpha[E_\delta(z)]}, \quad z \in H$$

for the averaging function of $\mu$ with respect to $A_\alpha$ and pseudohyperbolic $\delta$-disks. We also put

$$\|\hat{\mu}^{\alpha,\delta}\|_\infty := \sup_{z \in H} \hat{\mu}^{\alpha,\delta}(z),$$

which might be possibly $\infty$.

We recall the following well known characterizations (see [5] and references therein) for $0 < p < \infty$:

the embedding $A^p_\alpha(H) \subset L^p(d\mu)$ is bounded

$$\iff \|\hat{\mu}^{\alpha,\delta}\|_\infty < \infty \quad (2.7)$$

and

the embedding $A^p_\alpha(H) \subset L^p(d\mu)$ is compact

$$\iff \lim_{z \to \partial H} \hat{\mu}^{\alpha,\delta}(z) = 0. \quad (2.8)$$

Here, $\hat{H} := H \cup \{\infty\}$ and $\lim_{z \to \partial H} g(z) = 0$ means that $\sup_{H \setminus K} |g| \to 0$ as the compact set $K \subset H$ expands to the whole of $H$, or equivalently that $g(z) \to 0$ as $\text{Im} z \to 0$ and $g(z) \to 0$ as $|z| \to \infty$.

We say that $\mu$ is an $\alpha$-Carleson measure if either side of (2.7) holds. Also, we say that $\mu$ is a compact $\alpha$-Carleson measure if either side of (2.8) holds. Note that the notion of (compact) $\alpha$-Carleson measures is independent of the parameters $p$ and $\delta$.

2.4. Joint Pullback Measure. Given a positive Borel measure $\mu$ on $H$ and $\varphi \in S$, we denote by $\mu \circ \varphi^{-1}$ the pullback measure on $H$ defined by $(\mu \circ \varphi^{-1})(E) = \mu[\varphi^{-1}(E)]$ for Borel sets $E \subset H$. By a standard approximation argument we have the identity

$$\int_H (h \circ \varphi) d\mu = \int_H h d(\mu \circ \varphi^{-1}) \quad (2.9)$$

valid for Borel functions $h \geq 0$ on $H$.

Let $\alpha > -1$ and $0 < p < \infty$. The connection between composition operators and Carleson measures comes from (2.9). For example, it is immediate from (2.9) that $C_\varphi$ is bounded on $A^p_\alpha(H)$ if and only if $A_\alpha \circ \varphi^{-1}$ is an $\alpha$-Carleson measure. Moreover, for each $\delta \in (0, 1)$, one may verify

$$\|C_\varphi\|_{A^p_\delta(S)}^p \approx \|A_\alpha \circ \varphi^{-1}\alpha,\delta\|_\infty; \quad (2.10)$$

the constants suppressed above are independent of $\varphi$ and $p$. See [21, Section 7.2] for the disk setting. Also is well known via (2.9) that $C_\varphi$ is compact on $A^p_\alpha(H)$ if and only if $A_\alpha \circ \varphi^{-1}$ is a compact $\alpha$-Carleson measure.
Now, for \( \varphi, \psi \in S \), we define the joint pullback measure \( \omega_{\alpha,p;\varphi,\psi} \) by
\[
\omega_{\alpha,p;\varphi,\psi} := |p^\alpha(\varphi, \psi) dA_\alpha| \circ \varphi^{-1} + |p^\alpha(\varphi, \psi) dA_\alpha| \circ \psi^{-1}.
\] (2.11)
The role of this joint pullback measure lies in characterizing boundedness and compactness of the corresponding difference of composition operators as in the next lemma, which is taken from [5, Theorem 3.3].

**Lemma 2.2.** Let \( \alpha > -1 \), \( 0 < p < \infty \) and \( 0 < \delta < 1 \). For \( \varphi, \psi \in S \) the following assertions hold:

(a) \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(\mathbb{H}) \) if and only if \( \omega_{\alpha,p;\varphi,\psi} \) is an \( \alpha \)-Carleson measure. In this case
\[
\| \hat{\omega}_{\alpha,\delta,\varphi,\psi} \|_\infty \approx \| C_\varphi - C_\psi \|_{A^p_\alpha};
\]
the constants suppressed here depend only on \( \alpha \), \( p \) and \( \delta \).

(b) \( C_\varphi - C_\psi \) is compact on \( A^p_\alpha(\mathbb{H}) \) if and only if \( \omega_{\alpha,p;\varphi,\psi} \) is a compact \( \alpha \)-Carleson measure.

In connection with the equality (2.9) we also note for easier reference later that there is a constant \( C = C(\alpha, p, \delta) > 0 \) such that
\[
\int_{\mathbb{H}} |f|^p d\mu \leq C \int_{\mathbb{H}} |f|^p \hat{\mu}_{\alpha,\delta} dA_\alpha
\] (2.12)
for functions \( f \in A^p_\alpha(\mathbb{H}) \). In fact one may apply (2.6) in the left hand side of the above, interchange the order of integrations, and then conclude the asserted inequality by (2.4) and (2.5).

2.5. **Angular Derivative.** Before introducing angular derivatives in the half-plane setting, we need to clarify the notion of nontangential limits at boundary points of \( \hat{\mathbb{H}} \). Note \( \partial \hat{\mathbb{H}} = \mathbb{R} \cup \{ \infty \} \) where \( \mathbb{R} \) denotes the real line. Of course, the nontangential limit at a point in \( \mathbb{R} \) refers to the standard notion. Meanwhile, the nontangential limit at \( \infty \) refers to that associated with nontangential approach regions \( \Omega_\gamma \), \( \gamma > 0 \), consisting of all \( z \in \mathbb{C} \) such that \( \text{Im } z > \gamma |\text{Re } z| \). For a function \( h : \mathbb{H} \to \mathbb{H} \) and \( x \in \partial \hat{\mathbb{H}} \), we write \( h(x) = L \) (possibly \( \infty \)) if \( h \) has nontangential limit \( L \) at \( x \), i.e., \( \angle \lim_{z \to x} h(z) = L \).

The well-known notion of the angular derivatives on the unit disk \( \mathbb{D} \) (see [8, Section 2.3] or [18, Chapter 4]) can be translated into the half-plane analogues via the Cayley transformation
\[
\kappa(\zeta) := \frac{i + \zeta}{1 - \zeta}, \quad \zeta \in \mathbb{D},
\] (2.13)
which conformally maps \( \mathbb{D} \) onto \( \hat{\mathbb{H}} \). That is, given \( \varphi \in S \), we say that \( \varphi \) has finite angular derivative at \( x \in \partial \hat{\mathbb{H}} \) if the corresponding function
\[
\varphi_\kappa := \kappa^{-1} \circ \varphi \circ \kappa
\]
has finite angular derivative \( \varphi'_\kappa(\tilde{x}) \) at \( \tilde{x} := \kappa^{-1}(x) \in \partial \mathbb{D} \). In this case, \( \varphi \) is forced to have nontangential limit \( \varphi(x) \in \partial \hat{\mathbb{H}} \) at \( x \) by the corresponding property of \( \varphi_\kappa \) at \( \tilde{x} \).
Caused by the fact that \( x \) or \( \varphi(x) \) is possibly \( \infty \), explicit formulation of \( \varphi'_{\kappa}(\tilde{x}) \) involves complications as follows (see [5, Section 2.5]):

(i) For \( x, \varphi(x) \in \mathbb{R} \),

\[
\lim_{z \to x} \frac{\varphi(z) - \varphi(x)}{z - x} = \varphi'_{\kappa}(\tilde{x}) \left[ \frac{\varphi(x) + i}{x + i} \right]^2 ;
\]

(ii) For \( x \in \mathbb{R} \) and \( \varphi(x) = \infty \),

\[
\lim_{z \to x} \frac{1}{(z - x)\varphi(z)} = -\varphi'_{\kappa}(\tilde{x}) \left[ \frac{1}{x + i} \right]^2 ;
\]

(iii) For \( x = \infty \) and \( \varphi(\infty) \in \mathbb{R} \),

\[
\lim_{z \to \infty} z[\varphi(z) - \varphi(\infty)] = -\varphi'_{\kappa}(1) [\varphi(\infty) + i]^2 ;
\]

(iv) For \( x = \infty = \varphi(\infty) \),

\[
\lim_{z \to \infty} \frac{z}{\varphi(z)} = \varphi'_{\kappa}(1) .
\]

In each case of (i)-(iv) the limit in the left-hand side is naturally denoted by \( \varphi'(x) \). Note \( \varphi'(x) \neq 0 \) by the Julia-Carathéodory Theorem.

Elliott and Wynn obtained characterizations for bounded composition operators as in the theorem below. In fact they noticed the theorem in [10, Lemma 2.2 and Theorem 3.4] only for \( p = 2 \) with \( \|C_\varphi\|_{A_p^\alpha}^2 = [\varphi'(\infty)]^{\alpha+2} \), but remains true for all \( p \) by (2.10). In what follows and throughout the paper, we use the notation

\[
R_\varphi(z) := \frac{\text{Im } z}{\text{Im } \varphi(z)}
\]

for short.

**Theorem 2.3** (Elliott-Wynn). Let \( \alpha > -1 \) and \( 0 < p < \infty \). Then the following assertions are equivalent for \( \varphi \in \mathcal{S} \):

(a) \( C_\varphi \) is bounded on \( A_p^\alpha(\mathbb{H}) \);
(b) \( \varphi(\infty) = \infty \) and \( \varphi'(\infty) \) exists;
(c) \( \sup_{z \in \mathbb{H}} R_\varphi(z) < \infty \);
(d) \( \limsup_{z \to \infty} R_\varphi(z) < \infty \).

In this case quantities in (c) and (d) are both equal to \( \varphi'(\infty) \), and

\[
\|C_\varphi\|_{A_p^\alpha}^p \approx [\varphi'(\infty)]^{\alpha+2} ;
\]

the constants suppressed here depend only on \( \alpha \) and \( p \).

We will need the following fact later.
Lemma 2.4. Assume that $\varphi \in S$ has finite angular derivative at $x \in \partial \hat{H}$.
Then
$$\angle \lim_{z \to x} R_\varphi(z) = \begin{cases} 
|\varphi'(x)|^{-1} & \text{if } x, \varphi(x) \in \mathbb{R} \\
0 & \text{if } x \in \mathbb{R}, \varphi(x) = \infty \\
\infty & \text{if } x = \infty, \varphi(\infty) \in \mathbb{R} \\
\varphi'(\infty) & \text{if } x = \infty = \varphi(\infty).
\end{cases}$$

Proof. The identity
$$\frac{1 - |\zeta|^2}{1 - |\varphi(\zeta)|^2} = \left| \frac{\varphi(z) + i}{z + i} \right|^2 R_\varphi(z) \text{ where } \zeta := \kappa^{-1}(z)$$
is verified through a straightforward calculation. Note that the left-hand side of the above tends to $|\varphi'(\tilde{x})|^{-1}$ where $\tilde{x} = \kappa^{-1}(x)$ as $z \to x$ nontangentially by the Julia-Carathéodory Theorem. So, using the formulae in (i)-(iv), we conclude the lemma by another straightforward calculation. \hfill \Box

3. Characterizations

In this section we obtain characterizations for linear connectedness of two composition operators.

For $\varphi, \psi \in S$, consider the condition
$$\lim_{\rho(\varphi(z), \psi(z)) \to 1} [R_\varphi(z) + R_\psi(z)] = 0,$$
which is the half-plane analogue of Dai’s condition in [9]. Note from Theorem 2.3 that the condition above requires a certain type of Angular Derivative Cancellation property. So, we say that ADC holds for the pair $\{\varphi, \psi\}$ if (3.1) holds.

Remark 3.1. We remark in passing that ADC holds for the pair $\{\varphi, \psi\}$ if $C_\varphi - C_\psi$ is compact on $A_\alpha^p(\mathbb{H})$. In fact compactness of $C_\varphi - C_\psi$ on $A_\alpha^p(\mathbb{H})$ implies (see [5, Theorem 4.3])
$$\lim_{z \to \partial \hat{H}} [R_\varphi(z) + R_\psi(z)] \frac{\alpha+2}{\alpha} \rho(\varphi(z), \psi(z)) = 0 \quad (3.2)$$
which in turn implies (3.1). However, (3.1) does not imply (3.2) in general. For example, one may take $\varphi(z) := z + i$ and $\psi(z) := z + 2i$.

Next, we observe some fundamental properties related to ADC which were not noticed in [9]. The first property is that ADC is transitive.

Theorem 3.2. Let $\varphi, \psi, \tau \in S$. Suppose that ADC holds for the pairs $\{\varphi, \tau\}$ and $\{\tau, \psi\}$. Then ADC holds for the pair $\{\varphi, \psi\}$.

Proof. Before proceeding, we introduce a temporary notation:
$$\tilde{\rho}(z, w) := \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}, \quad z, w \in \mathbb{H}.$$
Suppose that ADC fails for the pair \{\varphi, \psi\}. So, there is a sequence \{z_k\} in \(H\) such that
\[
\rho(\varphi(z_k), \psi(z_k)) \to 1 \tag{3.3}
\]
and
\[
\inf_k [R_\varphi(z_k) + R_\psi(z_k)] > 0.
\]
By symmetry we may further assume that \(\text{Im } \psi \geq \text{Im } \varphi\) along the sequence \{z_k\} by passing to a subsequence if necessary. Thus we have
\[
\inf_k R_\varphi(z_k) > 0. \tag{3.4}
\]
Meanwhile, since ADC holds for the pair \{\varphi, \tau\} by assumption, we see from the above that
\[
\sup_k \rho(\varphi(z_k), \tau(z_k)) < 1. \tag{3.5}
\]
So, we obtain from (3.3) that
\[
\tilde{\rho}(\psi(z_k), \tau(z_k)) \geq \tilde{\rho}(\varphi(z_k), \psi(z_k)) - \tilde{\rho}(\varphi(z_k), \tau(z_k)) \to \infty
\]
and thus
\[
\rho(\psi(z_k), \tau(z_k)) \to 1. \tag{3.6}
\]
On the other hand, we see from (3.5) and (2.4) that \(\text{Im } \varphi(z_k) \approx \text{Im } \tau(z_k)\) and thus by (3.4)
\[
\inf_k R_\tau(z_k) > 0.
\]
This, together with (3.6), implies that ADC fails for the pair \{\tau, \psi\}, which is a contradiction. The proof is complete. \(\square\)

Next, we characterize ADC by means of Carleson measures. For that purpose, we need the next lemma.

\textbf{Lemma 3.3.} Given \(0 < \delta < 1\), there is a constant \(C = C(\delta) > 0\) such that
\[
\sup_{w \in E_\delta(z)} [1 - \rho(\varphi(w), \psi(w))] \leq C [1 - \rho(\varphi(z), \psi(z))]
\]
for all \(z \in H\) and \(\varphi, \psi \in S\).

\textbf{Proof.} Fix \(\delta \in (0, 1)\) and let \(\varphi, \psi \in S\). Let \(z \in H\). The asserted inequality is clear if \(\rho(\varphi(z), \psi(z)) \leq \frac{1}{2}\). So, assume \(\rho(\varphi(z), \psi(z)) > \frac{1}{2}\) for the rest of the proof. Let \(w \in E_\delta(z)\). Note that \(\varphi(w) \in E_\delta(\varphi(z))\) by (2.2). We thus have \(\text{Im } \varphi(w) \approx \text{Im } \varphi(z)\) by (2.4), and the same estimate also holds for \(\psi\).
It follows from (2.1) that
\[
\left[ 1 - \rho^2(\varphi(w), \psi(w)) \right]^{\frac{1}{2}} \approx \frac{\varphi(z) - \psi(z)}{\varphi(w) - \psi(w)} = \frac{1}{\rho(\varphi(z), \psi(z))} \frac{\varphi(z) - \psi(z)}{\varphi(w) - \psi(w)} \leq 2 \left| \frac{\varphi(z) - \psi(z)}{\varphi(w) - \psi(w)} \right|.
\]

Meanwhile, since \( \varphi(w) \in E_\delta(\varphi(z)) \), we have by (2.3)
\[
|\varphi(w) - \varphi(z)| < \frac{4\delta}{1 - \delta^2} \text{Im} \varphi(w) \leq \frac{4\delta}{1 - \delta^2} |\varphi(w) - \overline{\psi(w)}|
\]
and the same estimate also holds for \( \psi \). It follows that
\[
\left| \frac{\varphi(z) - \psi(z)}{\varphi(w) - \psi(w)} \right| \leq \frac{8\delta}{1 - \delta^2} + 1.
\]

So, combining these observations, we conclude the lemma. \( \square \)

In what follows \( \chi_E \) denotes the characteristic function of the set \( E \).

**Theorem 3.4.** Let \( \alpha > -1 \), \( 0 < p < \infty \) and \( 0 < \delta < 1 \). For \( \varphi, \psi \in S \) and \( 0 < r < 1 \), put
\[
\omega_r := (\chi_{G_r} dA_\alpha) \circ \varphi^{-1} + (\chi_{G_r} dA_\alpha) \circ \psi^{-1}
\]
where
\[
G_r := \{ z \in H : \rho(\varphi(z), \psi(z)) \geq r \}.
\]
Consider the following two assertions:

(a) ADC holds for the pair \( \{ \varphi, \psi \} \);
(b) \( \lim_{r \to 1} \| \tilde{\omega}_r^{\alpha,\delta} \|_\infty = 0 \).

Then the implication (b) \( \Rightarrow \) (a) holds. If, in addition, \( C_\varphi - C_\psi \) is bounded on some \( A_\beta^q(H) \) with \( \beta < \alpha \), then the implication (a) \( \Rightarrow \) (b) also holds.

**Proof.** Assume (b). We assume that (a) fails and seek a contradiction. As in the proof of Proposition 3.2, we may assume that there is a sequence \( \{ z_k \} \) in \( H \) such that
\[
\rho(\varphi(z_k), \psi(z_k)) \to 1
\]
and
\[
\inf \limits_k R_{\varphi}(z_k) =: \eta > 0. \quad (3.7)
\]
Using Lemma 3.3, pick a constant $C = C(\delta) > 0$ such that 
\[ \inf_{w \in E_\delta(z_k)} \rho(\varphi(w), \psi(w)) \geq 1 - C \left[ 1 - \rho(\varphi(z_k), \psi(z_k)) \right] =: r_k \]
and thus $E_\delta(z_k) \subset G_{r_k}^H$ for all large $k$. We also have by (2.2) that 
\[ E_\delta(z_k) \subset \varphi^{-1}(E_\delta(w_k)) \]
where $w_k := \varphi(z_k)$, for each $k$. Accordingly, we have 
\[ E_\delta(z_k) \subset \varphi^{-1}(E_\delta(w_k)) \cap G_{r_k} 
\]
and thus 
\[ A_\alpha[\varphi^{-1}(E_\delta(w_k)) \cap G_{r_k}] \geq A_\alpha[E_\delta(z_k)] \gtrsim \eta^{\alpha+2} A_\alpha[E_\delta(w_k)] \]
for all large $k$; the last estimate holds by (2.5) and (3.7). This yields 
\[ \hat{\omega}^{\alpha,\delta}(w_k) \gtrsim \eta^{\alpha+2} > 0 \]
for all large $k$ and this is a contradiction to (b).

We now assume (a) and prove (b) under the additional assumption that 
$C_{\varphi} - C_{\psi}$ is bounded on $A^q_\beta(H)$ for some $\beta < \alpha$ and $q$. For $\tau = \varphi$ or $\tau = \psi$, put 
\[ \epsilon_\tau(r) := \sup_{Gr} R_\tau \quad \text{and} \quad \omega_{\tau,\tau} := (\chi_\tau, dA_\alpha) \circ \tau^{-1} \]
for $0 < r < 1$. Note that $\epsilon_{\varphi}(r) + \epsilon_{\psi}(r)$ decreases to 0 as $r \to 1$ by assumption.

Also, note $\omega_\tau = \omega_{\tau,\varphi} + \omega_{\tau,\psi}$. Assume $\frac{1}{2} \leq r < 1$ for the rest of the proof.

Let $c := \alpha - \beta > 0$. We have
\[ \omega_{\tau,\tau}[E_\delta(z)] = A_\alpha[\tau^{-1}(E_\delta(z)) \cap G_r] \]
\[ = \int_{\tau^{-1}(E_\delta(z)) \cap G_r} R_\tau^c(w)[\text{Im} \tau(w)]^c dA_\beta(w) \]
\[ \leq \frac{c^c(r)}{r^q} \int_{\tau^{-1}(E_\delta(z))} [\text{Im} \tau(w)]^c \rho^q(\varphi(w), \psi(w)) dA_\beta(w) \]
\[ \approx \epsilon_\tau^c(r)(\text{Im } z)^c \int_{\tau^{-1}(E_\delta(z))} \rho^q(\varphi, \psi) dA_\beta \quad \text{by (2.4)} \]
\[ \approx \epsilon_\tau^c(r) \frac{A_\alpha[E_\delta(z)]}{A_\beta[E_\delta(z)]} \int_{\tau^{-1}(E_\delta(z))} \rho^q(\varphi, \psi) dA_\beta \quad \text{by (2.5)} \]
for all $z \in H$. Accordingly, we have by Lemma 2.2 
\[ \|\hat{\omega}^{\alpha,\delta}\|_\infty \lesssim [\epsilon_{\varphi}(r) + \epsilon_{\psi}(r)]^c \|C_{\varphi} - C_{\psi}\|^q_{A_\beta^q}; \]
the constant suppressed in this estimate is independent of $r$. So, taking the limit $r \to 1$, we conclude (b). The proof is complete. \[ \square \]

We now proceed to investigate the connection between ADC and linear connectedness of two composition operators. We first observe that ADC is a necessary condition for linear connectedness. We recall the following estimate, which is implicit in the proof of [5, Theorem 4.3].
Lemma 3.5. Given $\alpha > -1$ and $0 < p < \infty$, there is a constant $C = C(\alpha, p) > 0$ such that

$$\sup_{\mathcal{H}} \left[ (R_\varphi + R_\psi) \frac{\alpha + 2}{p} \rho(\varphi, \psi) \right] \leq C \| C_\varphi - C_\psi \|_{A^p_\alpha}$$

for all $\varphi, \psi \in \mathcal{S}$.

In what follows recall $\varphi_s = (1 - s)\varphi + s\psi$.

Proposition 3.6. Let $\alpha > -1$, $0 < p < \infty$ and $\varphi, \psi \in \mathcal{S}$. If $C_\varphi$ and $C_\psi$ are linearly connected in $\text{Comp}(A^p_\alpha)$, then ADC holds for the pair $\{\varphi, \psi\}$ and

$$\sup_{0 \leq s \leq 1} \| C_\varphi - C_{\varphi_s} \|_{A^p_\alpha} < \infty.$$ 

Proof. With Lemma 3.5 available, one may prove the first part by a straightforward modification of the proof of [9, Theorem 3.2] on the disk. Thus we omit the proof.

We now prove the second part. Suppose that $C_\varphi$ and $C_\psi$ are linearly connected in $\text{Comp}(A^p_\alpha)$. So, the linear path $\gamma : [0, 1] \to \text{Comp}(A^p_\alpha)$, defined by $\gamma(s) := C_{\varphi_s}$, is continuous and thus uniformly continuous. Therefore we can find a positive integer $N$ such that

$$\tilde{d}_{\alpha, p}(\gamma(s), \gamma(t)) \leq \frac{1}{2}, \quad \text{or equivalently,} \quad \| \gamma(s) - \gamma(t) \|_{A^p_\alpha} \leq 1$$

whenever $s, t \in [0, 1]$ and $|s - t| \leq \frac{1}{N}$. It follows that

$$\| \gamma(0) - \gamma(s) \|_{A^p_\alpha} \leq N$$

for all $s \in [0, 1]$. This completes the proof of the proposition. \qed

We observe some corollaries for easier reference later. The proof above actually shows the following.

Corollary 3.7. Let $\alpha > -1$, $0 < p < \infty$ and $\varphi, \psi \in \mathcal{S}$. If $C_\varphi$ and $C_\psi$ belong to the same path component of $\text{Comp}(A^p_\alpha)$, then $C_\varphi - C_\psi$ is bounded on $A^p_\alpha(H)$.

The next corollary is a consequence of Proposition 3.6 and Theorem 3.2.

Corollary 3.8. Let $\alpha > -1$, $0 < p < \infty$ and $\varphi, \psi \in \mathcal{S}$. If $C_\varphi$ and $C_\psi$ are polygonally connected in $\text{Comp}(A^p_\alpha)$, then ADC holds for the pair $\{\varphi, \psi\}$ and $C_\varphi - C_\psi$ is bounded on $A^p_\alpha(H)$.

We do not know in general whether the necessary conditions in Proposition 3.6 are also sufficient. However, under an additional assumption, they turn out to be also sufficient. To prove it, we need some preliminary estimates. The following is taken from [5, Lemma 3.5].

Lemma 3.9. Let $\alpha > -1$, $0 < p < \infty$ and $0 < \delta' < \delta < 1$. Then there is a constant $C = C(\alpha, p, \delta, \delta') > 0$ such that

$$|f(z) - f(w)|^p \leq C \frac{p^p(z, w)}{A_\alpha(E_\delta(w))} \int_{E_\delta(w)} |f|^p \, dA_\alpha$$
for all \( z, w \in H \) with \( z \in E_{\delta}(w) \) and functions \( f \) holomorphic on \( E_{\delta}(w) \).

See [9, Lemma 3.2] for the disk version of the following lemma, which was originally noticed in [12].

**Lemma 3.10.** For \( z, w \in H \) and \( s \in [0, 1] \), put \( z_s := (1 - s)z + sw \). Then

\[
\rho(z_s, z_t) \leq \frac{|s - t|\rho(z, w)}{1 - (|s - t|)\rho(z, w)}
\]

for all \( s, t \in [0, 1] \).

**Proof.** Fix \( z, w \in H \). First, note

\[
\rho(z_s, w) = \left| (1 - s)\rho(z, w) \right| \leq \frac{(1 - s)\rho(z, w)}{1 - s\rho(z, w)}
\]

for \( 0 \leq s \leq 1 \).

Now, let \( 0 \leq s \leq t \leq 1 \) and put \( \xi_s := z_{(1-s)(1-s)+s} \) for \( s' \in [0, 1] \). Note \( \xi_0 = w \), \( \xi_1 = z_s \) and \( \xi_{1-s} = z_t \). Thus, applying (3.8), we obtain

\[
\rho(z_s, z_t) = \rho \left( \xi_{1-s}, \xi_1 \right) \\
\leq \frac{(1 - \frac{1-t}{1-s})\rho(\xi_0, \xi_1)}{1 - \frac{1-t}{1-s}\rho(\xi_0, \xi_1)} \\
= \frac{(t-s)\rho(z_s, w)}{1 - s - (1 - t)\rho(z_s, w)}.
\]

So, applying (3.8) once more, we conclude the lemma by a little manipulation. The proof is complete. \( \square \)

See Remark 3.16 for motivation of the standing hypothesis (3.9) in the proposition below. The compactness parts of the proposition and its corollary are included for later use in Section 4. Also, with regard to statement (a) of the proposition, see Example 4.8 which shows that in general (3.9) does not imply \( \|C_\varphi - C_\psi\|_{A^p_\alpha} < \infty \).

**Proposition 3.11.** Let \( \alpha > \beta > -1 \) and \( 0 < p, q < \infty \). Let \( \varphi, \psi \in S \) and assume

\[
\sup_{0 \leq s \leq 1} \|C_\varphi - C_\psi_s\|_{A^q_\beta} < \infty. \quad \text{(3.9)}
\]

Then the following assertions hold:

(a) If \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \), then

\[
\sup_{0 \leq s \leq 1} \|C_\varphi - C_\psi_s\|_{A^p_\alpha} < \infty;
\]

(b) If \( C_\varphi - C_\psi \) is compact on \( A^p_\alpha(H) \), then so is \( C_\varphi - C_\psi_s \) for each \( s \in [0, 1] \).
Proof. Before proceeding, we introduce some temporary notation. Put
\[ M_{\beta,q} := \sup_{0 \leq s \leq 1} \| T_s \|_{A^{\beta}_q} \] where \( T_s := C_\varphi - C_{\varphi_s} \) for simplicity. Note \( M_{\beta,q} < \infty \) by assumption. Also, using the set \( G_r \) introduced in Theorem 3.4, put
\[ \eta(r) := \sup_{G_r} (R_\varphi + R_\psi) \] for simplicity. Note \( M_{\beta,q} < \infty \) by assumption. Also, using the set \( G_r \) introduced in Theorem 3.4, put
\[ \eta(r) := \sup_{G_r} (R_\varphi + R_\psi) \]
for \( \frac{1}{2} < r < 1 \). Since \( r \geq \frac{1}{2} \), we note
\[ \eta(r) \leq \frac{4}{3} C \eta(r) + 2 \] (3.10)
where \( C = C(\beta, q) > 0 \) is the constant provided by Lemma 3.5.

We now proceed to prove (a). Suppose that \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(\mathcal{H}) \). We fix \( r \in \left[ \frac{1}{2}, 1 \right) \) throughout the proof of (a). Let \( 0 \leq s \leq 1 \) and \( f \in A^p_\alpha(\mathcal{H}) \). We write
\[ \int_\mathcal{H} |T_s f|^p dA_\alpha = \int_{\mathcal{H}\setminus G_r} + \int_{G_r} \] (3.11)
and estimate the integrals in the right-hand side separately.

First, we consider the integral over \( G_r \) in (3.11). Note
\[ \rho(\varphi_s, \psi) = \frac{(1 - s) \rho(\varphi, \psi)}{1 - s \frac{\varphi - \psi}{\varphi - \psi}} \geq \frac{1 - s}{1 + s \rho(\varphi, \psi)} \geq \frac{1 - s}{2} \rho(\varphi, \psi) \] (3.12)
so that
\[ \rho(\varphi_s, \psi) \geq \frac{1 - s}{4} \text{ on } G_r \] (3.13)
for each \( s \). Accordingly, setting
\[ \mu_{s,r} = \mu_{s,\alpha,q,r} := \left[ \chi_{G_r} \rho^q(\varphi_s, \psi) dA_\alpha \right] \circ \varphi_s^{-1}, \]
we obtain
\[ (1 - s)^q \int_{G_r} |f(\varphi_s)|^p dA_\alpha \leq \int_{G_r} |f(\varphi_s)|^p \rho^q(\varphi_s, \psi) dA_\alpha \]
\[ = \int_\mathcal{H} |f|^p d\mu_{s,r} \]
\[ \leq \int_\mathcal{H} |f|^p \mu_{\alpha,\frac{q}{2}}^{\frac{1}{2}} dA_\alpha \] (3.14)
for all \( s \); the last estimate holds by (2.12).

We now pause to estimate \( \tilde{\mu}_{s,r}^{\alpha,\frac{1}{2}} \). Put
\[ E(w) := E_{\frac{1}{2}}(w), \quad w \in \mathcal{H} \]
for short. Since \( \text{Im } \varphi_s \geq \min\{\text{Im } \varphi, \text{Im } \psi\} \), we note \( R_{\varphi_s} \leq R_\varphi + R_\psi \) and thus by (2.4)
\[ \text{Im } z \leq \eta(r) \text{Im } \varphi_s(z) \approx \eta(r) \text{Im } w \]
for \( z \in \varphi_s^{-1}[E(w)] \cap G_r \). It follows that

\[
\mu_{s,r}[E(w)] = \int_{\varphi_s^{-1}[E(w)] \cap G_r} (\text{Im } z)^{\alpha-\beta} \rho^q(\varphi_s(z), \psi(z)) \, dA_\beta(z)
\]

\[
\lesssim [\eta(r)]^{\alpha-\beta} \int_{\varphi_s^{-1}[E(w)]} \rho^q(\varphi_s, \psi) \, dA_\beta
\]

\[
\approx [\eta(r)]^{\alpha-\beta} A_\alpha[E(w)] \int_{\varphi_s^{-1}[E(w)]} \rho^q(\varphi_s, \psi) \, dA_\beta;
\]

the last estimate holds by (2.5). Since this estimate holds for arbitrary \( w \), we obtain from Lemma 2.2 and (3.10) that

\[
\|\hat{\mu}_{s,r}\|_\infty \lesssim [\eta(r)]^{\alpha-\beta} M^q_{\beta,q} \lesssim M^q_{\beta,q} \quad (3.15)
\]

the constants suppressed here are independent of \( r \) and \( s \). Combining this with (3.14), we obtain

\[
(1-s)^q \int_{G_r} |f(\varphi_s)|^p \, dA_\alpha \lesssim M^q_{\beta,q} \|f\|_{A^p_\alpha}^p;
\]

which in turn yields

\[
(1-s)^q \int_{G_r} |T_s f|^p \, dA_\alpha \lesssim M^q_{\beta,q} \|f\|_{A^p_\alpha}^p
\]

(3.16) for all \( s \). Meanwhile, one may check

\[
\rho(\varphi, \varphi_s) \geq \frac{s}{4} \quad \text{on} \quad G_r
\]

as in the proof of (3.13). So, using

\[
\bar{\mu}_{s,r} = \varphi_s^{-1} \rho^q(\varphi, \varphi_s) \, dA_\alpha \circ \varphi_s^{-1}
\]

in place of \( \mu_{s,r} \) and repeating the same argument as above, one may verify

\[
s^q \int_{G_r} |T_s f|^p \, dA_\alpha \lesssim M^q_{\beta,q} \|f\|_{A^p_\alpha}^p
\]

(3.17) for all \( s \). Consequently, we deduce from (3.16) and (3.17)

\[
\int_{G_r} |T_s f|^p \, dA_\alpha \lesssim M^q_{\beta,q} \|f\|_{A^p_\alpha}^p
\]

(3.18) one may check that the constant suppressed in this estimate is independent of \( r, s \) and \( f \).

Now, we consider the integral over \( H \setminus G_r \) in (3.11). Fix an arbitrary \( s \in [0,1] \). Note

\[
\rho(\varphi, \varphi_s) \leq \rho(\varphi, \psi) < r \quad \text{on} \quad H \setminus G_r
\]

by Lemma 3.10. Thus, setting

\[
\bar{E}(w) := E_\delta(w) \quad \text{where} \quad \delta := \frac{1+r}{2}
\]
and applying Lemma 3.9 with \( \delta' = r \), we have
\[
|T_s f(z)|^p \lesssim \rho^p(\varphi(z), \varphi_*(z)) \int_{E(\varphi(z))} |f(w)|^p dA_\alpha(w) \tag{3.19}
\]
for \( z \in H \setminus G_r \). We also note from (2.5) and (2.4)
\[
A_\alpha[E(\varphi(z))] \approx A_\alpha[\overline{E}(w)], \quad w \in \overline{E}(\varphi(z)).
\]
This, together with (3.19), yields
\[
|T_s f(z)|^p \lesssim \rho^p(\varphi(z), \psi(z)) \int_{E(\varphi(z))} |f(w)|^p dA_\alpha(w) \tag{3.20}
\]
for \( z \in H \setminus G_r \). Now, integrating both sides of (3.20) over \( H \setminus G_r \) and then changing the order of integration, we obtain
\[
\int_{H \setminus G_r} |T_s f(z)|^p dA_\alpha(z)
\]
\[
\lesssim \int_H \left\{ \int_{\varphi^{-1}(E(w))} \rho^p(\varphi(z), \psi(z)) dA_\alpha(z) \right\} \frac{|f(w)|^p dA_\alpha(w)}{A_\alpha[\overline{E}(w)]}
\]
\[
\leq \int_H |f(w)|^p \omega^\alpha,\delta(w) dA_\alpha(w) \tag{3.21}
\]
where \( \omega := \omega_{\alpha,p;\varphi,\psi} \) is the joint pullback measure given in (2.11). Accordingly, we obtain by Lemma 2.2
\[
\int_{H \setminus G_r} |T_s f(z)|^p dA_\alpha(z) \lesssim \|C_\varphi - C_\psi\|_{\overline{A}_\alpha^p}^p \|f\|_{\overline{A}_\alpha^p}^p \tag{3.22}
\]
the constant suppressed in this estimate is also independent of \( s \) and \( f \). So, combining (3.22) with (3.18), we conclude
\[
\sup_{0 \leq s \leq 1} \|T_s\|_{\overline{A}_\alpha^p} \lesssim \|C_\varphi - C_\psi\|_{\overline{A}_\alpha^p} + M_{\beta,q}^{q(\alpha+2)/q(p+2)},
\]
as required. This completes the proof of (a).

We now turn to the proof of (b). Suppose that \( C_\varphi - C_\psi \) is compact on \( \overline{A}_\alpha^p(H) \). Fix \( s \in (0,1) \). To prove compactness of \( T_s \), we use Lemma 2.1. Let \( \{f_n\} \) be a sequence in \( \overline{A}_\alpha^p(H) \) such that \( \sup_n \|f_n\|_{\overline{A}_\alpha^p} \leq 1 \) and \( f_n \to 0 \) uniformly on each compact set of \( H \). We need to show
\[
\int_H |T_s f_n|^p dA_\alpha \to 0 \tag{3.23}
\]
for each \( s \in [0,1] \).

Given \( \frac{1}{2} \leq r < 1 \) and a compact set \( K \subset H \), we have by (3.21)
\[
\int_{H \setminus G_r} |T_s f_n|^p dA_\alpha \lesssim \int_{H \setminus K} |f_n|^p \omega^\alpha,\delta dA_\alpha
\]
\[
\leq \sup_{H \setminus K} \omega^\alpha,\delta + \left( \sup_K |f_n|^p \right) \|\omega^\alpha,\delta\|_\infty A_\alpha(K)
\]
for all \( n \); the constant suppressed above is independent of \( n \). Since \( \omega \) is a compact \( \alpha \)-Carleson measure by Lemma 2.2 and \( f_n \to 0 \) uniformly on \( K \), we see from the above that
\[
\lim_{n \to \infty} \int_{H \setminus G_r} |T_s f_n|^p dA_\alpha = 0
\]
for each \( r \). Meanwhile, using the first inequality in (3.15), we see from the proof of (a)
\[
\int_{G_r} |T_s f_n|^p dA_\alpha \lesssim \eta(r)^{\alpha-\beta} M^q_{\beta,q}
\]
for all \( n \) and \( r \). It follows that
\[
\limsup_{n \to \infty} \int_{H} |T_s f_n|^p dA_\alpha \lesssim \eta(r)^{\alpha-\beta} M^q_{\beta,q}
\]
for each \( r \); the constant suppressed here is independent of \( r \). Recall from Remark 3.1 that \( \eta(r) \to 0 \) as \( r \to 1 \). Consequently, taking the limit \( r \to 1 \), we conclude (3.23), as required. The proof is complete. \( \square \)

When \( \rho(\varphi, \psi) \) stays away from 1, the estimate for the integrals over \( G_r \) in the proof of Proposition 3.11 is vacuous for \( r \) sufficiently close to 1. Thus, as a by-product, we obtain the following corollary.

**Corollary 3.12.** Let \( \alpha > -1 \) and \( 0 < p < \infty \). Let \( \varphi, \psi \in S \) and assume
\[
\sup_{H} \rho(\varphi, \psi) < 1.
\]
Then the following assertions hold:

(a) If \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \), then
\[
\sup_{0 \leq s \leq 1} \|C_\varphi - C_\psi_s\|_{A^p_\alpha} < \infty;
\]

(b) If \( C_\varphi - C_\psi \) is compact on \( A^p_\alpha(H) \), then so is \( C_\varphi - C_\psi_s \) for each \( s \in [0, 1] \).

We are now ready to prove the converse of Proposition 3.6 under an additional assumption.

**Proposition 3.13.** Let \( \alpha > \beta > -1 \) and \( 0 < q, p < \infty \). Let \( \varphi, \psi \in S \) and assume (3.9). If ADC holds for the pair \( \{\varphi, \psi\} \) and \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \), then \( C_\varphi \) and \( C_\psi \) are linearly connected in \( \text{Comp}(A^p_\alpha) \).

**Proof.** We continue using the notation introduced in the proof of Proposition 3.11. Suppose that ADC holds for the pair \( \{\varphi, \psi\} \) and that \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \). First, we note from Proposition 3.11
\[
M_{p, p} := \sup_{0 \leq s \leq 1} \|C_\varphi - C_\psi_s\|_{A^p_\alpha} < \infty.
\]
Define $\gamma : [0, 1] \to \text{Comp}(A^p_\alpha)$ by $\gamma(s) := C_{\varphi_s}$. We prove the right continuity of $\gamma$; proof for the left continuity is similar. Fix $s_0 \in (0, 1)$ and let $s_0 \leq s < \frac{s_0 + 1}{2}$ for the rest of the proof. Put

$$T_{s_0,s} := C_{\varphi_{s_0}} - C_{\varphi_s}$$

for simplicity.

Let $\frac{1}{2} \leq r < 1$ and fix an arbitrary $f \in A^p_\alpha(H)$. By (3.12) we have

$$\rho(\varphi_s, \psi) \geq \frac{(1 - s_0) r}{4} \geq \frac{1 - s_0}{8} \quad \text{on} \quad G_r$$

for each $s$. Thus, using the first inequality in (3.15) and repeating the same argument as in the proof of Proposition 3.11, we obtain that

$$\int_{G_r} |T_{s_0,s} f|^p dA_\alpha \lesssim [\eta(r)]^{\alpha - \beta} M^q_{\beta,q} ||f||^p_{A^p_\alpha}; \quad (3.24)$$

the constant suppressed in this estimate is independent of $r, s$ and $f$.

Meanwhile, note from Lemma 3.10

$$\rho(\varphi_s, \varphi_{s_0}) \leq \rho(\varphi, \psi) < r \quad \text{on} \quad H \setminus G_r$$

and

$$\rho(\varphi_s, \varphi_{s_0}) \leq \frac{(s - s_0) \rho(\varphi, \psi)}{1 - r} \quad \text{on} \quad H \setminus G_r.$$ 

Thus, proceeding as in the proof of Proposition 3.11 and setting

$$\nu_{s_0} := [\rho^p(\varphi, \psi) dA_\alpha] \circ \varphi_{s_0}^{-1},$$

we obtain

$$\int_{H \setminus G_r} |T_{s_0,s} f|^p dA_\alpha \leq C_1 (s - s_0)^p \int_H |f|^p \varphi_{s_0}^\alpha \cdot \rho dA_\alpha \quad (3.25)$$

for some constant $C_1 = C_1(r, \alpha, p) > 0$ independent of $s$ and $f$. Moreover, since

$$\rho^p(\varphi, \psi) \leq c_\rho [\rho(\varphi, \varphi_{s_0}) + \rho(\varphi_{s_0}, \psi)]$$

where $c_\rho := \max\{1, 2^{p-1}\}$, we have by Lemma 2.2

$$||\varphi_{s_0}^{\alpha \cdot \rho}|| \leq C_2 \left( ||C \varphi - C \varphi_{s_0}||_{A^p_\alpha}^p + ||C \varphi_{s_0} - C \varphi||_{A^p_\alpha}^p \right) \leq (1 + 2c_\rho) C_2 M^p_{\alpha,p}$$

for some constant $C_2 = C_2(r, \alpha, p) > 0$ independent of $s$ and $f$; recall $\delta = \frac{1 + r}{2}$. This, together with (3.25), yields

$$\int_{H \setminus G_r} |T_{s_0,s} f|^p dA_\alpha \leq (1 + 2c_\rho) C_1 C_2 (s - s_0)^p M^p_{\alpha,p} ||f||^p_{A^p_\alpha}; \quad (3.26)$$

Now, we have by (3.24) and (3.26)

$$||T_{s_0,s}||^p_{A^p_\alpha} \lesssim [\eta(r)]^{\alpha - \beta} M^q_{\beta,q} + C_1 C_2 (s - s_0)^p M^p_{\alpha,p};$$

the constant suppressed in this estimate is independent of $r$ and $s$. So, taking first the limit $s \to s_0^+$ and then the limit $r \to 1$, we conclude that

$$\lim_{s \to s_0^+} ||T_{s_0,s}||_{A^p_\alpha} = 0.$$
Since $T_{s_0, s} = \gamma(s_0) - \gamma(s)$, this means that $\gamma$ is right continuous at $s_0$, as required. The proof is complete.

In what follows $\omega_r$ denotes the measure introduced in Theorem 3.4.

**Theorem 3.14.** Let $\alpha > \beta > -1$ and $0 < q, p < \infty$. Let $\varphi, \psi \in \mathcal{S}$ and assume that
\[ \|C_\varphi - C_\psi\|_{A^p_\alpha} < \infty \quad \text{and} \quad \sup_{0 \leq s \leq 1} \|C_\varphi - C_{\varphi_s}\|_{A^q_\beta} < \infty. \] (3.27)
Then the following assertions are equivalent:

(a) $C_\varphi$ and $C_\psi$ are linearly connected in $\text{Comp}(A^p_\alpha)$;
(b) $C_\varphi$ and $C_\psi$ are polygonally connected in $\text{Comp}(A^p_\alpha)$;
(c) ADC holds for the pair $\{\varphi, \psi\}$;
(d) $\lim_{r \to 1} \|\hat{\omega}^{\alpha, \delta}_r\|_{\infty} = 0$ for some/any $\delta \in (0, 1)$.

**Proof.** Clearly, (a) implies (b). The implication (b) $\implies$ (c) holds by Corollary 3.8. The implication (c) $\implies$ (a) holds by Proposition 3.13. So, (a), (b) and (c) are equivalent. Finally, (c) and (d) are equivalent by Theorem 3.4. \qed

**Remark 3.15.** In case $C_\varphi$ and $C_\psi$ are both bounded on some $A^q_\beta(H)$, the line segment $[C_\varphi, C_\psi]$ is uniformly bounded on any $A^p_\alpha(H)$ by Theorem 2.3. Thus the standing hypotheses (3.27) of Theorem 3.14 hold and statements (a) through (d) are equivalent.

With regard to the standing hypotheses in Propositions 3.11, 3.13 and Theorem 3.14, the following remark also seems worth mentioning.

**Remark 3.16.** In case
\[ \frac{\alpha + 2}{p} = \frac{\beta + 2}{q} \quad \text{with} \quad \beta < \alpha \quad \text{(and thus} \quad q < p), \] (3.28)
we remark that (3.9) implies boundedness of $C_\varphi - C_\psi$ on $A^p_\alpha(H)$. In fact, when (3.28) holds, we have
\[ \|C_\varphi - C_\psi\|_{A^p_\alpha} \leq C\|C_\varphi - C_\psi\|_{A^q_\beta} \]
for some constant $C = C(\alpha, \beta, p, q) > 0$; see [5, Corollary 4.6] and its proof.

We now consider the special case that $\rho(\varphi, \psi)$ stays away from 1. In this case, note that ADC trivially holds for the pair $\{\varphi, \psi\}$ and that the measures $\omega_r$ are trivial for all $r$ sufficiently close to 1.

**Theorem 3.17.** Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi, \psi \in \mathcal{S}$ and assume
\[ \sup_{\mathcal{H}} \rho(\varphi, \psi) < 1. \]
Then the following assertions are equivalent:

(a) $C_\varphi$ and $C_\psi$ are linearly connected in $\text{Comp}(A^p_\alpha)$;
(b) $C_\varphi$ and $C_\psi$ are polygonally connected in $\text{Comp}(A^p_\alpha)$;
(c) \( C_\varphi \) and \( C_\psi \) belong to the same path component of \( \text{Comp}(A^p_\alpha) \);
(d) \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \).

Proof. It is clear that (a) implies (b), which in turn implies (c). Also, (c) implies (d) by Corollary 3.7. Now suppose that (d) holds. Note from Corollary 3.12 that \( \sup_{0 \leq s \leq 1} \|C_\varphi - C_\psi\|_{A^p_\alpha} < \infty \). Moreover, since \( \rho(\varphi, \psi) \) stays away from 1, the set \( G_r \) used in the proof of Proposition 3.13 is empty for all \( r \) sufficiently close to 1. Thus, fixing such an \( r \), we see that (a) follows from (3.26). \( \square \)

Note that, when the images of \( \varphi \) and \( \psi \) are relatively compact in \( H \), \( \rho(\varphi, \psi) \) stays away from 1. In this regard, we recall the following fact taken from [5, Theorem 4.1] and its proof.

**Lemma 3.18.** Let \( \alpha > -1 \), \( 0 < p < \infty \) and \( \varphi, \psi \in S \). Assume that \( \varphi(H) \) and \( \psi(H) \) are contained in a compact set \( K \) in \( H \). Then the following assertions are equivalent:
(a) \( C_\varphi - C_\psi \) is compact on \( A^p_\alpha(H) \);
(b) \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \);
(c) \( \varphi - \psi \in A^p_\alpha(H) \).

In this case
\[
\|C_\varphi - C_\psi\|_{A^p_\alpha} \approx \|\varphi - \psi\|_{A^p_\alpha};
\]
the constants suppressed above depend only on \( \alpha \), \( p \) and \( K \).

Thus the next corollary is immediate from Theorem 3.17 and Lemma 3.18.

**Corollary 3.19.** Let \( \alpha > -1 \), \( 0 < p < \infty \) and \( \varphi, \psi \in S \). Assume that \( \varphi(H) \cup \psi(H) \) is compact in \( H \). Then the following assertions are equivalent:
(a) \( C_\varphi \) and \( C_\psi \) are linearly connected in \( \text{Comp}(A^p_\alpha) \);
(b) \( C_\varphi \) and \( C_\psi \) are polygonally connected in \( \text{Comp}(A^p_\alpha) \);
(c) \( C_\varphi \) and \( C_\psi \) belong to the same path component of \( \text{Comp}(A^p_\alpha) \);
(d) \( C_\varphi - C_\psi \) is compact on \( A^p_\alpha(H) \);
(e) \( C_\varphi - C_\psi \) is bounded on \( A^p_\alpha(H) \);
(f) \( \varphi - \psi \in A^p_\alpha(H) \).

Having seen Theorem 3.14, we are naturally led to the following question.

**Question 3.20.** Are polygonally connected composition operators necessarily linearly connected?

## 4. Applications and Examples

In this section we apply the results in the previous section to study the Shapiro-Sundberg question (stated in Introduction) and the (non-)isolation phenomena for the current setting. We also characterize when two composition operators induced by linear fractional self-maps belong to the same path component. In addition, we exhibit several related examples.
For the Shapiro-Sundberg question, we first introduce some notation. Given \( \alpha > -1 \), \( 0 < p < \infty \) and \( \varphi \in \mathcal{S} \), we denote by \( [C_{\varphi}]_{A^p_\alpha} \) the set of all composition operators \( C_\psi \) satisfying the following two conditions:

(i) \( C_{\varphi} - C_\psi \) is compact on \( A^p_\alpha(H) \);
(ii) \( \sup_{0 \leq s \leq 1} \|C_{\varphi} - C_{\varphi_s}\|_{A^q_\beta} < \infty \) for some \( \beta \in (-1, \alpha) \) and \( 0 < q < \infty \).

Also, put

\[ S_b := \{ \varphi \in \mathcal{S} : \varphi(\infty) = \infty \text{ and } \varphi'(\infty) \text{ exists} \} \]

By Theorem 2.3, \( S_b \) consists of all \( \varphi \in \mathcal{S} \) such that \( C_\varphi \) is bounded on some/any \( A^p_\alpha(H) \). Thus, for \( \varphi \in S_b \), Condition (ii) is redundant, again by Theorem 2.3.

In the theorem below, (a) holds by Proposition 3.11, and (b) holds by Remark 3.1 and Proposition 3.13.

**Theorem 4.1.** Let \( \alpha > -1 \), \( 0 < p < \infty \) and \( \varphi \in \mathcal{S} \). Then the following assertions hold for \( C_\psi \in [C_{\varphi}]_{A^p_\alpha} \):

(a) \( [C_{\varphi}, C_\psi] \subset [C_{\varphi}]_{A^p_\alpha} \);
(b) \( C_{\varphi} \) and \( C_\psi \) are linearly connected in \( \text{Comp}(A^p_\alpha) \).

In particular, \( [C_{\varphi}]_{A^p_\alpha} \) is path connected in \( \text{Comp}(A^p_\alpha) \). However, \( [C_{\varphi}]_{A^p_\alpha} \) might not be a whole path component of \( \text{Comp}(A^p_\alpha) \) in general. We provide two examples of such \( \varphi \); one is from \( S_b \) and the other from \( \mathcal{S} \setminus S_b \). Note that these examples also negate the Shapiro-Sundberg question in the current setting.

**Example 4.2.** Let \( \alpha > -1 \) and \( 0 < p < \infty \). Consider \( \varphi \in S_b \) given by

\[ \varphi(z) := z + 2i. \]

Then \( [C_{\varphi}]_{A^p_\alpha} \) is not a path component of \( \text{Comp}(A^p_\alpha) \).

**Proof.** We employ an auxiliary holomorphic function \( Q \) on \( H \) such that \( |Q| \leq 1 \) and \( Q(z) \to 1 \) as \( z \to \infty \). For example, one may take

\[ Q(z) := \frac{z}{z + i}. \]

Let \( a \) be an arbitrary complex number with \( 0 < |a| \leq 1 \) for the rest of the proof.

Clearly, we have

\[ \text{Im } \varphi \geq 2|Q| \quad \text{on } H \quad (4.1) \]

and thus

\[ \text{Im } (\varphi + aQ) \geq \text{Im } \varphi - |a||Q| \geq \frac{1}{2} \text{Im } \varphi \quad \text{on } H. \]

This, together with Theorem 2.3, implies \( \varphi + aQ \in S_b \). Moreover, we have

\[ \rho(\varphi, \varphi + aQ) = \frac{|a||Q|}{2\text{Im } \varphi - aQ} \leq \frac{|a|}{2\text{Im } \varphi - |a|} \leq \frac{1}{3} \quad \text{on } H; \]
the last inequality holds by (4.1). We thus deduce by Theorem 3.17 that $C_\varphi$ and $C_{\varphi + aQ}$ belong to the same path component of $\text{Comp}(A_\alpha^p)$. Now, in order to complete the proof, it suffices to show that $C_{\varphi + aQ} \notin [C_\varphi]_{A_\alpha^p}$. Note $\text{Im} \varphi = 3$ on the horizontal line $i + \mathbb{R}$. We thus have

$$\rho(\varphi, \varphi + aQ) R_\alpha^p = \frac{|a||Q|}{3 \alpha^2 |6i - aQ|} \quad \text{on} \quad i + \mathbb{R}.$$ 

It follows that

$$\rho(\varphi(z), \varphi(z) + aQ(z)) R_\alpha^p (z) \rightarrow \frac{|a|}{3 \alpha^2 |6i - a|} > 0$$

as $z \rightarrow \infty$ along $i + \mathbb{R}$. Thus, $C_\varphi - C_{\varphi + aQ}$ is not compact on $A_\alpha^p(\mathbb{H})$ by Remark 3.1 and, consequently, $C_{\varphi + aQ} \notin [C_\varphi]_{A_\alpha^p}$, as asserted. The proof is complete.  

Example 4.3. Let $\alpha > -1$ and $0 < p < \infty$. Consider $\varphi \in S \setminus S_b$ given by

$$\varphi(z) := 2\pi i + \log(z + ei).$$

Then $[C_\varphi]_{A_\alpha^p}$ is not a path component of $\text{Comp}(A_\alpha^p)$. 

Proof. This time our auxiliary function is

$$Q(z) := \frac{\pi}{(z + ei)^\lambda} \quad \text{with} \quad \lambda := \frac{\alpha + 2}{p}.$$ 

Note $|Q| < \pi$ on $\mathbb{H}$ and thus $\text{Im} (\varphi + Q) > \pi$ on $\mathbb{H}$. So, $\varphi + Q \in S$. It is known that $C_{\varphi} - C_{\varphi + Q}$ is bounded on $A_\alpha^p(\mathbb{H})$, but not compact; see [5, Example 7.9]. Thus, $C_{\varphi + Q} \notin [C_\varphi]_{A_\alpha^p}$. 

Meanwhile, since $\text{Im} \varphi \geq 2|Q|$ on $\mathbb{H}$, we have

$$\rho(\varphi, \varphi + Q) \leq \frac{1}{3} \quad \text{on} \quad \mathbb{H}$$

as in the proof of Example 4.2. Hence, it follows from Theorem 3.17 that $C_\varphi$ and $C_{\varphi + Q}$ belong to the same path component of $\text{Comp}(A_\alpha^p)$. The proof is complete.  

We now turn to the investigation of sufficient conditions for (non-)isolation of composition operators. For non-isolation, we note from Corollary 3.19 that $C_\varphi$ with $\varphi(\mathbb{H})$ compact in $\mathbb{H}$ is not isolated in any $\text{Comp}(A_\alpha^p)$. In fact, given such $\varphi$, one may pick $Q \in A_\alpha^p(\mathbb{H})$ with $|Q| \leq 1$ and $\epsilon > 0$ sufficiently small in order that $\varphi + \epsilon Q \in S$ has relatively compact image in $\mathbb{H}$. So, $C_\varphi$ and $C_{\varphi + \epsilon Q}$ are linearly connected in $\text{Comp}(A_\alpha^p)$ by Corollary 3.19. The next theorem is an extension of this observation.

**Theorem 4.4.** Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi \in S$ and assume that there is a holomorphic function $Q$ on $\mathbb{H}$ satisfying the following two conditions:

- (i) $\text{Im} \varphi \geq |Q|$ on $\mathbb{H}$;
Then $C_\varphi$ is not isolated in $\text{Comp}(A_\alpha^p)$.

Proof. Fix a complex number $a$ with $|a| \leq \frac{1}{2}$ and put $\psi := \varphi + aQ$. By (i), $\text{Im} \psi \geq \text{Im} \varphi - \frac{|Q|}{2} \geq \frac{1}{2} \text{Im} \varphi$ and thus $\psi \in S$. It suffices to show that $C_\varphi$ and $C_\psi$ are linearly connected in $\text{Comp}(A_\alpha^p)$.

Let $f \in A_\alpha^p(H)$. Since $\text{Im} \varphi_r \geq \frac{1}{2} \text{Im} \varphi$ for $0 \leq r \leq 1$, we have by (2.6) and the Cauchy Estimates

$$\sup_{0 \leq r \leq 1} |f'(\varphi_r)| \lesssim \frac{\|f\|_{A_\alpha^p}}{(\text{Im} \varphi)^{\frac{\alpha + 2 + p}{2} + 1}}; \quad (4.2)$$

for convenience. The following is an easy consequence of Theorem 4.4.

**Corollary 4.5.** Let $\alpha > -1$, $0 < p < \infty$ and $\varphi \in S_h$. Then $C_\varphi$ is not isolated in $\text{Comp}(A_\alpha^p)$.

Proof. Assume $\inf_H (\text{Im} \varphi) =: \delta > 0$ and pick $Q \in A_\alpha^p(H)$ with $|Q| \leq \delta$. For example, one may take

$$Q(z) := \delta \left( \frac{1}{z + i} \right)^{\frac{\alpha + 2 + \sigma}{p}} \text{ with } \sigma > 0. \quad (4.3)$$

Thus the corollary holds by Theorem 4.4.

In what follows we put

$$S_h := \left\{ \varphi \in S : \inf_H (\text{Im} \varphi) > 0 \right\}$$

for convenience. The following is an easy consequence of Theorem 4.4.

**Corollary 4.6.** Let $\alpha > -1$, $0 < p < \infty$ and $\varphi, \psi \in S_h$. If $\varphi - \psi \in A_\alpha^p(H)$, then $C_\varphi$ and $C_\psi$ are linearly connected in $\text{Comp}(A_\alpha^p)$. \qed
Proof. Note $|\varphi_s - \varphi_t| = |s - t||\varphi - \psi|$ for $s, t \in [0, 1]$. Thus, since $\varphi, \psi \in S_h$ by assumption, the same argument as in the proof of Theorem 4.4 yields
\[
\|C_{\varphi_s} - C_{\varphi_t}\|_{A^p_\alpha} \leq C\|\varphi - \psi\|_{A^p_\alpha}|s - t|, \quad s, t \in [0, 1]
\]
for some constant $C > 0$ independent of $s$ and $t$. This implies the corollary.
\[\Box\]

In view of Corollary 4.5, one may ask what happens if the inducing map touches the finite boundary. In this regard, we pause to construct a non-isolated composition operator whose inducing map touches a finite boundary point.

For $\lambda > 0$ and $0 < \eta < 1$, consider functions $\varphi^{\lambda, \eta} \in S$ defined by
\[
\varphi^{\lambda, \eta}(z) := i \left[1 - \left(\frac{i}{z + i}\right)\right]^\eta.
\]
Before proceeding, we observe some basic properties of these functions. First, since
\[
\text{Re} \left[1 - \left(\frac{i}{z + i}\right)^\lambda\right] \geq 1 - \frac{1}{|z + i|^\lambda} > 0
\]
for $z \in H$, we see that $\varphi^{\lambda, \eta}(H)$ is contained in a cone in $H$ (depending on $\eta$) with vertex at the origin. Thus
\[
\text{Im} \varphi^{\lambda, \eta} \geq C|\varphi^{\lambda, \eta}| \quad \text{on } H
\]
for some constant $C = C(\eta) > 0$. Next, since
\[
\varphi^{\lambda, \eta}(0) = 0 \quad \text{and} \quad \lim_{z \to \infty} |\varphi^{\lambda, \eta}(z) - i||z + i|^\lambda = \eta
\]
we have
\[
|\varphi^{\lambda, \eta}(z) - i| \approx \frac{1}{|z + i|^\lambda}
\]
for all $z \in H$. Finally, since
\[
\lim_{z \to 0} \frac{|\varphi^{\lambda, \eta}(z)|}{|z|^\eta} = \lambda^\eta \quad \text{and} \quad \lim_{z \to \infty} \varphi^{\lambda, \eta}(z) = i,
\]
we obtain
\[
|\varphi^{\lambda, \eta}(z)| \approx \begin{cases} |z|^\eta & \text{for } z \in D \\ 1 & \text{for } z \in H \setminus D \end{cases}
\]
where $D := \{z \in H : |z| < 1\}$.

Note that $\varphi^{\lambda, \eta}$ always touches a finite boundary point, i.e., $\varphi^{\lambda, \eta}(0) = 0$. We see in the example below that, given $\alpha$ and $p$, a suitably-chosen $\varphi^{\lambda, \eta}$ induces a composition operator which is not isolated in $\text{Comp}(A^p_\alpha)$.
Example 4.7. Let $\lambda > 0$ and $0 < \eta < 1$. For $\alpha > -1$ and $0 < p < \infty$, define $\gamma_{\lambda, \eta} : [0, 1] \rightarrow \text{Comp}(A^p_\alpha)$ by

$$
\gamma_{\lambda, \eta}(s) := C_{\varphi^\lambda, \eta} \quad \text{where} \quad \varphi^\lambda_{s, \eta} := (1 - s)\varphi^\lambda_{s, \eta} + si, \quad 0 \leq s \leq 1.
$$

Then the following assertions hold:

(a) $\gamma_{\lambda, \eta}$ is continuous on $(0, 1]$ if and only if $\lambda > \alpha + \frac{2}{p}$;

(b) $\gamma_{\lambda, \eta}$ is continuous on $[0, 1]$ if $\lambda > \alpha + \frac{2}{p}$ and $0 < \eta < \frac{\alpha + 2}{\alpha + 2 + p}$.

Proof. Put $\varphi := \varphi^\lambda_{s, \eta}$ and $\gamma := \gamma_{\lambda, \eta}$ for simplicity. First, we prove (a). Fix $s_0 \in (0, 1]$. Let $s$ be sufficiently close to $s_0$ so that $s_0 - \frac{s_0}{2} < s < 1 + s_0$. Note that $\varphi_s(H)$ is contained in a compact set $K \subset H$ depending only on $s_0$. Thus we have by Lemma 3.18 that

$$
\|\gamma(s_0) - \gamma(s)\|_{A^p_\alpha} \approx \|\varphi_{s_0} - \varphi_s\|_{A^p_\alpha} = |s_0 - s| \|\varphi_i\|_{A^p_\alpha};
$$

the constants suppressed in this estimate depend only on $\alpha$, $p$ and $s_0$. Note from (4.5)

$$
\|\varphi - i\|_{A^p_\alpha} < \infty \iff \lambda > \frac{\alpha + 2}{p}. \tag{4.7}
$$

Thus (a) holds.

We now prove (b). By (a) we only need to consider continuity at $s = 0$. So, let $s$ be sufficiently close to $0$, say $0 < s < \frac{1}{2}$, and consider an arbitrary $f \in A^p_\alpha(H)$. In order to estimate $\|C_{\varphi_s} - C_{\varphi}\|_{A^p_\alpha}$, we write

$$
\int_H |f(\varphi_s(z)) - f(\varphi(z))|^p dA_\alpha(z) = \int_D + \int_{H \setminus D} \tag{4.8}
$$

where $D$ is the set specified in (4.6).

Before proceeding, we first estimate

$$
M_s(z) := \sup_{0 \leq t \leq s} |f'(\varphi_t(z))|
$$

for $z \in H$. Note, for $0 \leq t \leq s$,

$$
\text{Im} \varphi_t = (1 - t)\text{Im} \varphi + t \geq (1 - s)\text{Im} \varphi \geq \frac{1}{2} \text{Im} \varphi \geq |\varphi|;
$$

the last inequality holds by (4.4). This, together with (2.6) and the Cauchy Estimates, yields

$$
M_s(z) \lesssim \frac{\|f\|_{A^p_\alpha}}{|\varphi(z)|^{\alpha + 2 + 1}};
$$

the constant suppressed here depends only on $\alpha$, $p$ and $\eta$. Accordingly, we have by (4.6)

$$
M_s(z) \lesssim \begin{cases} 
\|f\|_{A^p_\alpha} |z|^{-\eta(\frac{\alpha + 2}{p} + 1)} & \text{for } z \in D \\
\|f\|_{A^p_\alpha} & \text{for } z \in H \setminus D.
\end{cases} \tag{4.9}
$$
We are now ready to estimate the integrals in the right-hand side of (4.8). Note
\[ |f(\varphi_s(z)) - f(\varphi(z))| \leq M_s(z) |\varphi_s(z) - \varphi(z)| = s M_s(z) |\varphi(z) - i| \]  
(4.10)
for \( z \in H \). Since \( |\varphi - i| \approx 1 \) on \( D \) by (4.5), we have by (4.10) and (4.9)
\[
\int_D |f(\varphi_s(z)) - f(\varphi(z))|^p \, dA_\alpha(z) \lesssim s^p \| f \|_{A^p_\alpha}^p \int_D \frac{dA_\alpha(z)}{|z|^\eta(\alpha + 2 + p)}
\]
\[
= C s^p \| f \|_{A^p_\alpha}^p \int_0^1 \frac{r^{\alpha + 1} dr}{r^\eta(\alpha + 2 + p)}
\]
where \( C = \int_0^\pi (\sin \theta)\alpha \, d\theta \). We also have by (4.10), (4.9) and (4.5)
\[
\int_{H \cap D} |f(\varphi_s(z)) - f(\varphi(z))|^p \, dA_\alpha(z) \lesssim s^p \| f \|_{A^p_\alpha}^p \int_{H \cap D} \frac{dA_\alpha(z)}{|z + i|^{\lambda \beta}}
\]
Accordingly, we obtain
\[
\|C_{\varphi_s} - C_{\varphi}\|_{A^p_\beta}^p \lesssim s^p \left\{ \int_0^1 \frac{r^{\alpha + 1} dr}{r^\eta(\alpha + 2 + p)} + \int_{H \cap D} \frac{dA_\alpha(z)}{|z + i|^{\lambda \beta}} \right\}.
\]
One may keep track of the constants suppressed above to find them independent of \( s \). Now, for \( \eta < \frac{\alpha + 2}{\alpha + 2 + p} \) and \( \lambda > \frac{\alpha + 2}{p} \), the integrals above are both finite and thus \( \|\gamma(s) - \gamma(0)\|_{A^p_\beta} = \|C_{\varphi_s} - C_{\varphi}\|_{A^p_\beta} \to 0 \) as \( s \to 0 \), as required. So, (b) holds. The proof is complete. \( \square \)

We observe that the functions \( \varphi^{\lambda,\eta} \) have an additional property related to Proposition 3.11 and Remark 3.16. In what follows the subscript \( i \) in \( C_i \) denotes the constant function \( i \).

**Example 4.8.** Let \( \alpha > \beta > -1 \) and \( 0 < p, q < \infty \) with \( \frac{\beta + 2}{q} < \frac{\alpha + 2}{p} \). Let \( \frac{\beta + 2}{q} < \lambda \leq \frac{\alpha + 2}{p} \), and \( 0 < \eta < 1 \). Then, for each \( s \in (0, 1) \),
\[
\sup_{0 < t \leq 1} \|C_{\varphi^{\lambda,\eta}_t} - C_i\|_{A^p_\beta} < \infty \quad \text{but} \quad \|C_{\varphi^{\lambda,\eta}_s} - C_i\|_{A^p_\beta} = \infty. \]  
(4.11)

**Proof.** Fix \( 0 < s < 1 \). Note that \( \varphi^{\lambda,\eta}(H) \) is contained in a compact subset of \( H \). Also, note \( \varphi^{\lambda,\eta} - i = (1 - s)(\varphi^{\lambda,\eta} - i) \). Thus (4.11) holds by Lemma 3.18 and (4.7). \( \square \)

The next question after Example 4.7 might be what happens if the inducing map touches the boundary more frequently. For example, consider an automorphism \( \varphi \), which is probably the most extreme case. When \( \varphi \) does not fix \( \infty \), it is known that \( C_{\varphi} \) cannot form a bounded difference with any other composition operator; see [5, Theorem 6.2]. Also, when \( \varphi \) fixes \( \infty \), one may use (the first part of) Proposition 3.6 to deduce that \( C_{\varphi} \) cannot be linearly connected to any other composition operator. These observations suggest that a composition operator tends to be isolated if its inducing map touches the boundary too often.
Now, in order to quantify “touching the boundary too often”, we use the notion of angular derivatives. To this end we introduce some notation and terminology. Put

\[ F(\varphi) := \{ x \in \partial \hat{H} : \varphi \text{ has finite angular derivative at } x \} \]

for \( \varphi \in S \). Recall that, at each \( x \in F(\varphi) \), \( \varphi \) has nontangential limit \( \varphi(x) \in \partial \hat{H} \) and \( \varphi'(x) \neq 0 \). The ordered pair \((\varphi(x), \varphi'(x))\) is naturally referred to as the first-order data of \( \varphi \) at \( x \in F(\varphi) \).

We now proceed to find a sufficient condition for isolation in terms of angular derivatives. To begin with, we recall the next two preliminary lemmas, taken from [5, Lemma 5.2] and [5, Lemma 5.3], respectively.

**Lemma 4.9.** Let \( \varphi, \psi \in S \) and \( x \in F(\varphi) \cap F(\psi) \). Assume that \( \varphi \) and \( \psi \) do not have the same first-order data at \( x \). Then

\[
\lim_{\epsilon \to 0^+} \lim_{z \to x} \frac{\varphi(z) - \varphi(z)}{z - x} = 0.
\]

Here, \( L_{\epsilon,x} := \{ z \in \mathbb{H} : \Im z = \epsilon \Re (z - x) \} \) for \( x \in \mathbb{R} \) and \( L_{\epsilon,\infty} := \{ z \in \mathbb{H} : \Im z = -\epsilon \Re z \} \).

**Lemma 4.10.** Let \( \varphi, \psi \in S \) and \( x \in F(\varphi) \setminus F(\psi) \) with \( \varphi(x) \in \mathbb{R} \). Then

\[
\angle \lim_{z \to x} \frac{\varphi(z) - \varphi(z)}{\varphi(z) - \psi(z)} = 0.
\]

Applying these two lemmas, we obtain the next lemma which might be of its own interest.

**Lemma 4.11.** Let \( \varphi, \psi \in S \). Let \( x \in F(\varphi) \) with either \( \varphi(x) \in \mathbb{R} \) or \( x = \infty = \varphi(\infty) \). Assume that \( \varphi \) and \( \psi \) do not have the same first-order data at \( x \). Then, given \( \alpha > -1 \) and \( 0 < p < \infty \),

\[
\|C_\varphi - C_\psi\|_{A^p_\alpha} \geq C \times \begin{cases} \|\varphi'(x)\|^{-\frac{\alpha+2}{p}} & \text{if } x, \varphi(x) \in \mathbb{R} \\ \infty & \text{if } x = \infty, \varphi(\infty) \in \mathbb{R} \\ [\varphi'(\infty)]^{-\frac{\alpha+2}{p}} & \text{if } x = \infty = \varphi(\infty) \end{cases}
\]

for some constant \( C > 0 \) depending only on \( \alpha \) and \( p \).

**Proof.** Fix a number \( \sigma > 0 \) and consider the functions

\[
k_w(z) := \frac{(\Im w)^{\sigma}}{(z - w)^{\frac{\alpha+2+\sigma}{p}}} \in A^p_\alpha(H)
\]
for \( w \in H \). By an elementary calculation (or see [5, Lemma 2.5]) one may check that \( \|k_w\|_{A^p_0} \) is independent of \( w \). It follows that

\[
\|C_\varphi - C_\psi\|_{A^p_0} \geq \|(C_\varphi - C_\psi)k_{\varphi(z)}\|_{A^p_0} \\
\geq (\operatorname{Im} z)^{\frac{\alpha + 2}{p}} |k_{\varphi(z)}(\varphi(z)) - k_{\varphi(z)}(\psi(z))| \quad \text{by (2.6)} \\
= 2^{\frac{\alpha + 2 + \sigma}{p}} k_{\varphi} \left( z \right) 1 - \left( \frac{\varphi(z) - \varphi(z)}{\psi(z) - \varphi(z)} \right)^{\frac{\alpha + 2 + \sigma}{p}}
\]

(4.12)

for all \( z \in H \); the constants suppressed above depend only on \( \alpha, p \) and \( \sigma \).

First, consider the case \( x = \infty = \varphi(\infty) \). Since \( \infty \in F(\varphi) \), we have \( \varphi \in S_b \) by Theorem 2.3. Thus, to avoid triviality, we may also assume \( \psi \in S_b \) so that \( \infty \in F(\varphi) \cap F(\psi) \). Thus we conclude the asserted inequality by Lemma 4.9 and Lemma 2.4.

Next, consider the case \( \varphi(x) \in R \). There are two possibilities; (i) \( x \in F(\psi) \) and (ii) \( x \notin F(\psi) \). In Case (i), applying Lemma 4.9, we obtain by (4.12) and Lemma 2.4

\[
\|C_\varphi - C_\psi\|_{A^p_0} \geq C \times \begin{cases} |\varphi'(x)|^{-\frac{\alpha + 2}{p}} & \text{if } x \in R \\ \infty & \text{if } x = \infty \end{cases}
\]

for some constant \( C = C(\alpha, p, \sigma) > 0 \). In Case (ii), since \( \varphi(x) \in R \), we also obtain the same estimate by Lemma 4.10. The proof is complete. \( \square \)

In what follows \( |\cdot| \) denote the Lebesgue measure on \( R \).

**Theorem 4.12.** Let \( \alpha > -1 \) and \( 0 < p < \infty \). Let \( \varphi \in S \) and assume \( |F(\varphi) \cap R| > 0 \). Then \( C_\varphi \) is isolated in \( \operatorname{Comp}(A^p_0) \).

**Proof.** To begin with, we recall the well-known fact that associated with each \( \tau \in S \) is its boundary function, still denoted by \( \tau \), defined almost everywhere on \( R \) by finite nontangential limits. Moreover, if the boundary functions of two maps in \( S \) match on a set of positive measure, then they match on the whole \( H \). Such properties are readily transferred from the unit disk through the Caley transformation (2.13).

Consider \( \psi \in S \) with \( \psi \neq \varphi \). Set \( N_\psi := \{x \in R : \varphi(x) = \psi(x)\} \) and \( F_\psi := [F(\varphi) \cap R] \setminus [\varphi^{-1}(\infty) \cup N_\psi] \). By the aforementioned remark we have \( |\varphi^{-1}(\infty) \cap R| = |N_\psi| = 0 \). So, we obtain

\[
\|C_\varphi - C_\psi\|_{A^p_0} \geq C \left( \sup_{x \in F_\psi} \frac{1}{|\varphi'(x)|} \right)^{\frac{\alpha + 2}{p}} = C \left\| \frac{1}{\varphi'} \right\|_{L^\infty(F(\varphi) \cap R)}^{\frac{\alpha + 2}{p}}
\]

where \( C = C(\alpha, p) > 0 \) is the constant provided by Lemma 4.11. This completes the proof. \( \square \)

As an immediate consequence of Theorem 4.12, we see that composition operators induced by automorphisms are all isolated. It turns out that,
among linear fractional self-maps of $H$, automorphisms are only ones with such property. We put

$$S_{LF} := \{ \varphi \in S : \varphi \text{ is a linear fractional map} \}$$

for convenience. As is quite elementary, $S_{LF}$ contains $\text{Aut}(H)$, the set of all automorphisms of $H$.

**Theorem 4.13.** Let $\varphi \in S_{LF}$. Then $C_\varphi$ is isolated in some/any $A_0^p(H)$ if and only if $\varphi \in \text{Aut}(H)$.

**Proof.** Note $F(\varphi) = \partial \hat{H}$ for $\varphi \in \text{Aut}(H)$. Thus the sufficiency is immediate from Theorem 4.12.

We now prove the necessity. So, assume $\varphi \notin \text{Aut}(H)$. We need to show that $C_\varphi$ is not isolated in any $A_0^p(H)$.

Consider Case (ii). Note that $\varphi^{-1}(R)$ consists of exactly one point, say $x_0 \in \partial \hat{H}$. Let $r$ be the radius of $\varphi(H)$ so that the center is $\varphi(x_0) + ir$. Then we have $|\varphi - \varphi(x_0) - ir| \leq r$, i.e.,

$$|\varphi - \varphi(x_0)|^2 \leq 2r \text{Im } \varphi \text{ on } H. \quad (4.13)$$

Now, given $\alpha > -1$ and $0 < p < \infty$, pick $h \in A_0^p(H)$ with $|h| \leq 1$ and put

$$Q := [\varphi - \varphi(x_0)]^{(2\alpha + 2) + 2} h;$$

note that $\varphi - \varphi(x_0)$ is zero-free on $H$. Note from (4.13)

$$|Q|^p \leq (2r)^{\alpha + 2 + p} |h|^p \text{ on } H.$$

Also, since $|\varphi - \varphi(x_0)| \leq 2r$, we have

$$|Q| \leq (2r)^{\frac{2(\alpha + 2)}{p} + 1} \text{Im } \varphi \text{ on } H.$$

We thus conclude by Theorem 4.4 that $C_\varphi$ is not isolated in $\text{Comp}(A_0^p)$, as desired. The proof is complete. $\square$

We notice below another application of Lemma 4.11 in a different direction. In the context of the space of composition operators acting on the weighted Bergman spaces over the unit disk, it has been long known (for $p = 2$) that two composition operators cannot belong to the same path component unless the inducing maps have exactly the same first-order data; see [14, Theorem 2.4]. Such a property does not extend to the half-plane setting. For example, consider $\varphi, \psi \in S_b$ defined by

$$\varphi(z) := z + i - \frac{1}{z} \quad \text{and} \quad \psi(z) := z + i - \frac{2}{z}.$$

Composition operators induced by these maps form a compact difference on any $A_0^p(H)$, but do not have the same first-order data at 0; see [5, Example
However, by Theorem 4.1, they belong to the same path component with respect to any \( \text{Comp}(A_0^p) \).

It turns out that the pathology mentioned in the preceding paragraph cannot happen when one of the two inducing maps is restricted to a certain subclass of \( S \), namely,

\[
S_\ell := \{ \varphi \in S : \varphi(x) \in \mathbb{R} \text{ for all } x \in F(\varphi) \cap \mathbb{R} \}.
\]

**Theorem 4.14.** Let \( \alpha > -1 \) and \( 0 < p < \infty \). Let \( \varphi \in S_\ell \) and \( \psi \in S \). Assume that \( C_\varphi \) and \( C_\psi \) belong to the same path component of \( \text{Comp}(A_0^p) \). Then the following assertions hold:

(a) \( F(\varphi) \subset F(\psi) \);

(b) \( \varphi \) and \( \psi \) have the same first-order data at each point in \( F(\varphi) \).

If, in addition, \( \psi \in S_\ell \), then \( F(\varphi) = F(\psi) \).

**Proof.** The second part being immediate from (a) by symmetry, we only need to prove the first part. Let \( x \in F(\varphi) \). Since \( \varphi \in S_\ell \), we have either \( \varphi(x) \in \mathbb{R} \) or \( x = \infty = \varphi(\infty) \). Let \( C = C(\alpha, p) > 0 \) be the constant provided by Lemma 4.11 and put

\[
\epsilon = \epsilon(x, \alpha, p) := C |\varphi'(x)|^{\pm \frac{\alpha+2}{p}} > 0
\]

where \( \pm \) is to be determined by whether \( \varphi(x) \in \mathbb{R} \) or not.

Let \( \gamma : [0, 1] \to \text{Comp}(A_0^p) \) be a continuous path such that \( \gamma(0) = C_\varphi \) and \( \gamma(1) = C_\psi \). As in the proof of Proposition 3.6, pick a positive integer \( N \) such that

\[
\|\gamma(s) - \gamma(t)\|_{A_0^p} < \epsilon
\]

whenever \( |s - t| \leq \frac{1}{N} \) and \( s, t \in [0, 1] \). Pick \( \tau_j \in S \) such that \( \gamma(\frac{j}{N}) = C_{\tau_j} \) for \( j = 1, 2, \ldots, N-1 \). Also, put \( \tau_0 := \varphi \) and \( \tau_N := \psi \). We then have by (4.14)

\[
\|C_{\tau_j-1} - C_{\tau_j}\|_{A_0^p} < \epsilon
\]

for each \( j = 1, \ldots, N \). It follows from the choice of \( \epsilon \) and Lemma 4.11 that

\[
x \in F(\tau_{j-1}) \cap F(\tau_j) \quad \text{and} \quad (\tau_{j-1}(x), \tau_j'(x)) = (\tau_j(x), \tau_j'(x))
\]

for each \( j = 1, \ldots, N \), successively. This implies (a) and (b). The proof is complete. \( \square \)

As an application of Theorem 4.14, we will describe below exactly when two composition operators induced by linear fractional maps belong to the same path component. Our characterization is quite different from that in the setting of the disk, where such operators belong to the same path component if and only if the inducing maps have the same first-order data. This can be seen by extending [9, Theorem 3.3] to general \( p \) through a direct estimate as in [2] (or through the joint Carleson measure arguments of the current paper) and then applying [9, Propositions 3.1 and 4.4].
In view of Theorem 4.13, we consider linear fractional self-maps which are not automorphisms. So, let

$$S^*_{\text{LF}} := S_{\text{LF}} \setminus \text{Aut}(H)$$

for convenience. For $\varphi \in S^*_{\text{LF}}$, note from Theorem 2.3 that $C_{\varphi}$ is bounded on $A^p_\alpha(H)$ if and only if $\varphi(\infty) = \infty$. Also, note $F(\varphi) = \varphi^{-1}(\partial H) \cap \partial H$ contains at most one point, because $\varphi$ is not an automorphism.

For $\varphi \in S^*_{\text{LF}}$, we use the notation

$$D\varphi(\infty) := \lim_{z \to \infty} z[\varphi(z) - \varphi(\infty)].$$

This limit always exists and is equal to $\varphi'(\infty)$ if $\varphi(\infty) \in \mathbb{R}$. The next lemma is taken from [6, Theorems 3.9–3.10]. Recall that $S_h$ denotes the class specified in (4.3).

**Lemma 4.15.** Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi, \psi \in S^*_{\text{LF}} \cap S_h$ be distinct maps such that $\varphi(\infty) \neq \infty \neq \psi(\infty)$. Then the following assertions are equivalent:

(a) $C_{\varphi} - C_{\psi}$ is bounded on $A^p_\alpha(H)$;
(b) $C_{\varphi} - C_{\psi}$ is compact on $A^p_\alpha(H)$;
(c) $\varphi(\infty) = \psi(\infty) \in H$ and one of the following two conditions is fulfilled:
   (i) $D\varphi(\infty) \neq D\psi(\infty)$ and $p > \alpha + 2$;
   (ii) $D\varphi(\infty) = D\psi(\infty)$ and $p > \frac{\alpha + 2}{2}$.

We now obtain a characterization of when composition operators induced by distinct maps in $S^*_{\text{LF}}$ belong to the same path component.

**Theorem 4.16.** Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi, \psi \in S^*_{\text{LF}}$ be distinct maps. Then the following assertions are equivalent:

(a) $C_{\varphi}$ and $C_{\psi}$ are linearly connected in $\text{Comp}(A^p_\alpha)$;
(b) $C_{\varphi}$ and $C_{\psi}$ belong to the same path component of $\text{Comp}(A^p_\alpha)$;
(c) $\varphi, \psi \in S_h$ and one of the following three conditions is fulfilled:
   (i) $\varphi(z) = az + b$ and $\psi(z) = az + c$ for some $a > 0$ and $b, c \in H$ with $b \neq c$;
   (ii) $\varphi(\infty) = \psi(\infty) \in H$, $D\varphi(\infty) \neq D\psi(\infty)$ and $p > \alpha + 2$;
   (iii) $\varphi(\infty) = \psi(\infty) \in H$, $D\varphi(\infty) = D\psi(\infty)$ and $p > \frac{\alpha + 2}{2}$.

**Proof.** That (a) implies (b) is trivial. We now prove that (b) implies (c). Assume that $C_{\varphi}$ and $C_{\psi}$ belong to the same path component of $\text{Comp}(A^p_\alpha)$. Then $C_{\varphi} - C_{\psi}$ is bounded on $A^p_\alpha(H)$ by Corollary 3.7. So, in case neither $\varphi$ nor $\psi$ fixes $\infty$, we have

$$\varphi(\infty) = \psi(\infty) =: \xi$$

by [6, Lemma 3.2]. This remains valid by Theorem 2.3 even when $\varphi$ or $\psi$ fixes $\infty$.

In case $\xi \in \mathbb{R}$, note that $\varphi(\mathbb{R})$ and $\psi(\mathbb{R})$ are circles tangent to $\mathbb{R}$ at $\xi$; recall $\varphi, \psi \notin \text{Aut}(H)$. So, $C_{\varphi} - C_{\psi}$ is not bounded on $A^p_\alpha(H)$ by [6, Theorem 3.3], which is impossible.
In case $\xi = \infty$, note that $\varphi$ and $\psi$ are linear polynomials. Also, note $F(\varphi) = F(\psi) = \{\infty\}$ and thus $\varphi, \psi \in S_f$. So, by Theorem 4.14, $\varphi$ and $\psi$ have the same first-order data at $\infty$. It follows that $\varphi$ and $\psi$ are of the form as in (i).

Let $\xi \in H$ for the rest of the proof. First, assume $\varphi, \psi \in S_f$. By Theorem 4.14 there are two possibilities:

- $F(\varphi) = F(\psi) = \emptyset$;
- $F(\varphi) = F(\psi) = \{x\}$ for some $x \in \mathbb{R}$.

In the former case, $\varphi(H)$ and $\psi(H)$ are both closed disks in $H$. So, $\varphi, \psi \in S_h$ and thus (ii) or (iii) holds by Lemma 4.15. In the latter case, note that $\varphi(x), \psi(x) \in \mathbb{R}$. Thus $(\varphi(x), \varphi'(x)) = (\psi(x), \psi'(x))$ by Theorem 4.14. Now, since $\varphi$ and $\psi$ are completely determined by the triple $(\xi, \varphi(x), \varphi'(x)) = (\xi, \psi(x), \psi'(x))$, we conclude $\varphi = \psi$, which is not possible by assumption.

Next, assume either $\varphi \notin S_f$ or $\psi \notin S_f$. Note by Theorem 4.14 that neither $\varphi(R)$ nor $\psi(R)$ intersects $R$. Also, note that each of $\varphi(R)$ and $\psi(R)$ is either a circle or a straight line parallel to $R$. Accordingly, we have $\varphi, \psi \in S_h$ and thus (ii) or (iii) holds by Lemma 4.15. So, we conclude (c).

Finally, we prove that (c) implies (a). To begin with, assume $\varphi, \psi \in S_h$. If (i) holds, then $C_\varphi$ and $C_\psi$ are both bounded on $A_{\beta}(H)$ by Theorem 2.3, and $\rho(\varphi, \psi)$ stays away from 1. So, (a) holds by Theorem 3.17 in Case (i).

Now, assume that (ii) or (iii) holds. Since $\xi \in H$, $\varphi$ and $\psi$ are of the form

$$\varphi(z) = \xi + \frac{D\varphi(\infty)}{z - v} \quad \text{and} \quad \psi(z) = \xi + \frac{D\psi(\infty)}{z - w}$$

for some $v, w \in \mathbb{H}$ or $\mathbb{R}$. So, choosing

$$r > \max\{|x| : x \in F(\varphi) \cup F(\psi)|,$$

which is regarded as 0 if $F(\varphi) = F(\psi) = \emptyset$, we have

$$|\varphi(z) - \psi(z)| \lesssim \frac{1}{|z|}, \quad z \in H \setminus rD$$

where $D$ is the set introduced in (4.6). This yields

$$\int_{H \setminus rD} |\varphi - \psi|^q dA_\beta < \infty$$

whenever $q > \beta + 2$. For such $q$ and $\beta$, we may repeat the argument in the proof of Corollary 4.6 to obtain

$$\int_{H \setminus rD} |f(\varphi) - f(\varphi_s)|^q dA_{\beta} \lesssim \|f\|^q_{A_\beta^q} \int_{H \setminus rD} |\varphi - \psi|^q dA_{\beta}; \quad (4.15)$$

the constant suppressed above is independent of $f \in A_\beta^q(H)$ and $s \in [0, 1]$.

Meanwhile, since $\Im \varphi_s \geq \min\{\Im \varphi, \Im \psi\} \gtrsim 1$, we have by a similar argument

$$\int_{rD} |f(\varphi_s)|^q dA_{\beta} \lesssim \|f\|^q_{A_\beta^q} A_{\beta}(rD)$$
and thus
\[ \int_{rD} |f(\varphi) - f(\varphi_s)|^q \, dA_\beta \lesssim \int_{rD} \|f(\varphi)|^q + |f(\varphi_s)|^q \, dA_\beta \lesssim \|f\|_{A_\beta^q}^q \beta_d(rD); \]

the constants suppressed above are also independent of \( f \in A_\beta^q(H) \) and \( s \in [0,1] \). Now, combining this with (4.15), we obtain
\[ \sup_{0 \leq s \leq 1} \|C_\varphi - C_{\varphi_s}\|_{A_\beta^q} < \infty \text{ whenever } q > \beta + 2. \tag{4.16} \]
Furthermore, \( C_\varphi - C_\psi \) is compact on \( A_\beta^p(H) \) by Lemma 4.15. Consequently, we conclude (a) by (4.16) and Theorem 4.1. This completes the proof. □

As a consequence of Theorem 4.16, unlike the disk case, we see that two linear fractional maps can induce composition operators belonging to the same path component, but may fail to have the same first-order data. For example, the functions
\[ \varphi(z) := i - \frac{1}{z}, \quad \psi(z) := i - \frac{2}{z + i}, \quad \tau(z) := i - \frac{3}{z - 1} \]
all belong to the same path component of \( \text{Comp}(A_\alpha^p) \) for \( p > \alpha + 2 \).

Examples of continuous paths we have considered up to now are all line segments. We now close the paper with examples of non-polygonal continuous paths.

**Example 4.17.** Let \( \varphi \in S_{LF}^* \) be given by
\[ \varphi(z) := a + b \left( \frac{z - w}{z - \overline{w}} \right) \]
where \( w \in H \) and \( a, b \in \mathbb{C} \) with \( \text{Im } a > |b| > 0 \). Define
\[ \gamma_\varphi(s) := C_{\Phi_s} \quad \text{where} \quad \Phi_s(z) := \varphi(sz) \]
for \( 0 < s \leq 1 \). Then \( \gamma_\varphi : (0,1] \to \text{Comp}(A_\alpha^p) \) is continuous for \( p > \alpha + 2 \).

**Proof.** Assume \( p > \alpha + 2 \) and let \( f \in A_\alpha^p(H) \). Let \( s \in (0,1] \) and \( \frac{s}{2} < t < 1 \). Setting
\[ h_s(z) := \sup_{\frac{s}{2} \leq \lambda \leq 1} |\varphi'(\lambda z)| = \sup_{\frac{s}{2} \leq \lambda \leq 1} \frac{2|b| \text{Im } w}{|\lambda z - \overline{w}|^2}, \]
note
\[ |\Phi_s(z) - \Phi_t(z)| \leq |s - t||z| h_s(z) \]
for \( z \in H \). Note \( \text{Im } \Phi_\lambda \geq \text{Im } a - |b| > 0 \) for all \( \lambda \in [0,1] \). We thus have by (4.2)
\[ |f(\Phi_s(z)) - f(\Phi_t(z))| \lesssim |s - t||z| h_s(z) \|f\|_{A_\alpha^p} \quad (4.17) \]
for \( z \in H \). Note that \( h_s \) is bounded by \( \frac{2|b|}{\text{Im} w} \) on \( H \). Also, note

\[
h_s(z) \leq 2|b|\text{Im} w \left( \frac{4}{s|z|} \right)^2 \quad \text{for} \quad |z| \geq \frac{4|w|}{s}.
\]

So, since \( p > \alpha + 2 \), we see that

\[
\int_H |z|^p|h_s(z)|^p \, dA_\alpha < \infty.
\]

This, together with (4.17), implies

\[
\|C_{\Phi_s} - C_{\Phi_t}\|_{\mathcal{A}_p^\alpha} \lesssim |s - t|, \quad \frac{s}{2} < t < 1;
\]

the constant suppressed here does not depend on \( t \). This shows that \( \gamma_\phi \) is continuous at \( s \), as required. The proof is complete. \( \square \)

One may easily modify the proof of Example 4.17 to produce examples of similar type. For example, take \( \varphi \in \mathcal{S} \) of the form

\[
\varphi(z) := a + b \left( \frac{\text{Im} w}{z - \bar{w}} \right)^\eta
\]

where \( w \in H \), \( \eta > 0 \), and \( a, b \in \mathbb{C} \) with \( \text{Im} a > |b| > 0 \). Then \( \gamma_\varphi : (0, 1] \to \text{Comp}(\mathcal{A}_p^n) \), defined as above, is continuous for \( p > \frac{a+2}{\eta} \). Also, there are examples of different type. For example, let

\[
\Phi_{1,s}(z) := i - \frac{1 + s}{z + (1 + s)i} \quad \text{and} \quad \Phi_{2,s}(z) := i - \frac{1}{z + (1 + s)i}
\]

for \( 0 \leq s \leq 1 \). Then, one may verify that the corresponding path \( s \mapsto C_{\Phi_{j,s}} \) is continuous from \( [0, 1] \) into \( \text{Comp}(\mathcal{A}_p^n) \) for \( p > \frac{a+2}{j} \).

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