COMPACT LINEAR COMBINATION OF COMPOSITION OPERATORS ON BERGMAN SPACES

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ABSTRACT. Motivated by the question of Shapiro and Sundberg raised in 1990, study on linear combinations of composition operators has been a topic of growing interest. In this paper, we completely characterize the compactness of any finite linear combination of composition operators with general symbols on the weighted Bergman spaces in two classical terms: one is a function theoretic characterization of Julia-Carathéodory type and the other is a measure theoretic characterization of Carleson type. Our approach is completely different from what have been known so far.

1. INTRODUCTION

Let $S = S(D)$ be the class of all holomorphic self-maps of the unit disk $D$ of the complex plane $C$. Each $\varphi \in S$ induces a composition operator $C_\varphi : H(D) \to H(D)$ defined by

$$C_\varphi f := f \circ \varphi,$$

where $H(D)$ is the class of all holomorphic functions on $D$. An extensive study on the theory of composition operators has been established during the past four decades on various settings. We refer to standard references [7] and [25] for various aspects on the theory of composition operators acting on holomorphic function spaces.

We first recall our function spaces to work on. Let $dA$ be the area measure on $D$ normalized to have total mass 1. For $\alpha > -1$, put

$$dA_\alpha(z) := (\alpha + 1)(1 - |z|^2)^\alpha dA(z), \quad z \in D;$$

the constant $\alpha + 1$ is chosen so that $A_\alpha(D) = 1$. Now, for $0 < p < \infty$, the $\alpha$-weighted Bergman space $A_\alpha^p(D)$ is the space of all $f \in H(D)$ such that the “norm”

$$\|f\|_{A_\alpha^p} = \left\{ \int_D |f(z)|^p dA_\alpha(z) \right\}^{1/p}$$

is finite. As is well-known, the space $A_\alpha^p(D)$ equipped with the norm above is a Banach space for $1 \leq p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|^p_{A_\alpha^p}.$

As is well known in the setting of $D$, every composition operator is bounded on well-known function spaces such as the weighted Bergman spaces $A_\alpha^p(D)$ and the Hardy spaces $H^p(D)$ due to the Littlewood Subordination Principle; see [7] or [25] for precise definition.
of the Hardy spaces $H^p(D)$. So, boundedness on those spaces is out of question and much efforts have been expended in the early stage on characterizing those maps in $S$ which induce compact composition operators. An early result of Shapiro and Taylor [27] in 1973 showed that the Julia-Carathéodory type condition

$$R_\varphi(z) := \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \to 0 \quad \text{as} \quad |z| \to 1 \quad (1.1)$$

is necessary for $\varphi \in S$ to induce a compact composition operator on $H^2(D)$ (and hence for the general Hardy spaces $H^p(D)$). This means via the Julia-Carathéodory Theorem that the non-existence of the angular derivative of the inducing map at any boundary point is a necessary condition for the compactness of a composition operator on the Hardy spaces. However, (1.1) turned out to be not sufficient. In fact, later in 1987 Shapiro [24] completely characterized the compactness of composition operators on the Hardy spaces by finding the precise formula for the essential norm of a composition operator on $H^2(D)$ in terms of the Nevanlinna counting function. The situation for the weighted Bergman spaces turned out to be quite different from the Hardy space case. Namely, MacCluer and Shapiro [18] proved that (1.1) is a necessary and sufficient condition for the compactness of composition operators on the weighted Bergman spaces.

At the same time, MacCluer [16] noticed that Carleson measure, which first appeared in the solution of the corona problem, is an important tool in the study of composition operators. Even though it is usually not easy to verify that a given measure satisfies the Carleson measure criteria, in most cases it is through this process one verifies that a composition operator is bounded (in case of higher dimensional setting) or compact. In fact, the connection between a single composition operator and a Carleson measure comes from the standard identity (see [9, p. 163])

$$\int_D |f \circ \varphi(z)|^p dA_\alpha(z) = \int_D |f(z)|^p d(A_\alpha \circ \varphi^{-1})(z) \quad (1.2)$$

valid for functions $\varphi \in S$ and $f \in H(D)$; see Section 2.4 for precise definition of the pullback measure $A_\alpha \circ \varphi^{-1}$. Due to the change-of-variable formula (1.2) one can easily deduce the Carleson measure criterion for the compactness of a single composition operator on the weighted Bergman spaces. That is, $C_\varphi$ is compact on $A^p_\alpha(D)$ if and only if $A_\alpha \circ \varphi^{-1}$ is a compact $\alpha$-Carleson measure on $A^p_\alpha(D)$; see Section 2.4 for the notion of $\alpha$-Carleson measures. This Carleson measure criterion, which is independent of $p$, play a fundamental role in the study of composition operators on the weighted Bergman spaces.

With the basic questions such as boundedness and compactness settled, more attention has been paid to the study of the topological structure of the composition operators in the operator norm topology, which is one of continuing interests in the theory of composition operators. In 1981 Berkson [1] initiated the study of the topological structure with his isolation result on the Hardy spaces. Berkson’s isolation result was refined later by Shapiro and Sundberg [26], and also by MacCluer [17]. In [26] Shapiro and Sundberg posed a question on whether two composition operators belong to the same connected component, when their difference is compact. While this question was originally for the Hardy spaces, it also initiated similar study on various other settings including the weighted Bergman spaces. It was answered negatively on both the Hardy spaces (see [2, 8, 20]) and the weighted Bergman spaces (see [19]).

The aforementioned question of Shapiro and Sundberg initiated another direction of study, i.e., the study of compact differences of composition operators on various settings, which has been a very active topic. While the characterization for compact differences still
remains open in the Hardy space case, it is completely settled in the weighted Bergman space case. More explicitly, Moorhouse [19] characterized the compactness of $C_\varphi - C_\psi$ on $A^2_{\alpha}(D)$ by the Julia-Carathéodory type condition

$$M_{\varphi,\psi}(z) := [R_{\varphi}(z) + R_{\psi}(z)]\rho_{\varphi,\psi}(z) \to 0 \quad \text{as} \quad |z| \to 1,$$

where $\rho_{\varphi,\psi} := \rho(\varphi,\psi)$. Here, $\rho$ denotes the pseudohyperbolic distance on $D$; see Section 2.2 for the definition of $\rho$. We remark in passing that this characterization has been extended not only to higher dimensional balls and polydisks, but also to general parameter $p$; see [3, 4]. The essence of Moorhouse’s characterization is that suitable cancellations should occur at every boundary point which makes either one of the inducing maps fail to induce a compact composition operator. For further results on compact differences on various other settings, we refer to [10, 11, 12, 13, 17, 20, 21, 22, 23] and references therein.

With the lack of the change-of-variable formula for the difference of two composition operators, Koo and Wang [13] introduced a new notion of joint Carleson measures to obtain a Carleson criterion for differences of two composition operators on $A^p_\alpha(B)$, the weighted Bergman space over the ball $B$, to be bounded/compact. As a consequence of their result, it turns out that the bounded/compact differences on $A^p_\alpha(B)$ depend on the index $p$, when the dimension is bigger than 1. This is in sharp contrast with the one dimensional case; note that Moorhouse’s characterization (1.3) is independent of $p$. Meanwhile, in case each composition operator is bounded on $A^p_\beta(B)$ for some $-1 < \beta < \alpha$, the compact difference on $A^p_\alpha(B)$ is known to be independent of $p$; see [3] and, for a similar result on the polydisk, [4]. We also remark that the compact difference on $H^p(D)$, $p \geq 1$, is also known to be independent of $p$; see [21].

Along the same line of study on differences, study on linear combinations has been a topic of growing interest. Kriete and Moorhouse [15] first obtained some general, but not complete, results on compact linear combinations on $A^2_{\alpha}(D)$. More recently, Koo and Wang [14] obtained complete characterizations for compact linear combinations of three composition operators on $A^p_\alpha(D)$. The current authors [6] characterized compact double differences on $A^p_\alpha(D)$. In this paper, in the setting of the weighted Bergman spaces $A^p_\alpha(D)$, we completely characterize the compact linear combinations of composition operators in two directions which have been discussed so far; one is the Julia-Carathéodory type characterization and the other is the Carleson criterion by means of joint Carleson measures. Our approach, dealing with general linear combinations, is completely different from what have been known so far.

We introduce several notation to be used throughout the paper. Given an arbitrary (but fixed) integer $N \geq 2$, we reserve symbol functions

$$\varphi_1, \varphi_2, \ldots, \varphi_N \in S$$

and coefficients

$$a_1, \ldots, a_N \in \mathbb{C}\{0\}$$

to be considered throughout the paper. We put

$$T_j := C_{\varphi_j} \quad \text{and} \quad T_{j,k} := T_j - T_k$$

for $j, k = 1, \ldots, N$. We put

$$T := \sum_{j=1}^N a_j T_j$$

for short.
Let $\mathcal{P}_N$ be the group of all permutations on the index set

$$\Lambda_N := \{1, 2, \ldots, N\}.$$  

Identifying $\sigma \in \mathcal{P}_N$ with the ordered $N$-tuple $(\sigma_1, \ldots, \sigma_N)$ where $\sigma_j := \sigma(j)$, we will sometimes write

$$\sigma = (\sigma_1, \ldots, \sigma_N).$$

Given $\sigma \in \mathcal{P}_N$, put

$$s_\sigma^j = s_\sigma^j(a_1, \ldots, a_N) := \sum_{k=1}^j a_\sigma_k$$

for $j \in \Lambda_N$. Note that we may represent $T$ as

$$T = \sum_{j=1}^{N-1} s_\sigma^j T_{\sigma_j, \sigma_{j+1}} + s_N^\sigma T_{\sigma_N}$$

(1.4)

for each $\sigma \in \mathcal{P}_N$.

To state our first main result we introduce more notation. We put for simplicity

$$R_j := R_{\varphi_j}, \quad M_{j,k} := M_{\varphi_j, \varphi_k} \quad \text{and} \quad \rho_{j,k} := \rho_{\varphi_j, \varphi_k}$$

for $j,k \in \Lambda_N$. Note that each $R_j$, and thus each $M_{j,k}$ as well, is bounded on $D$ by the Schwarz-Pick Lemma. Now, motivated by the representation (1.4), we define

$$(1.5) \quad Q_\sigma := \sum_{j=1}^{N-1} |s_\sigma^j| M_{\sigma_j, \sigma_{j+1}} + |s_N^\sigma| R_{\sigma_N}$$

for $\sigma \in \mathcal{P}_N$ and put

$$(1.6) \quad Q := \prod_{\sigma \in \mathcal{P}_N} Q_\sigma.$$  

Note that functions $Q_\sigma$ are all bounded on $D$.

Our first result is the Julia-Carathéodory type characterization below. In what follows, $\lambda_z$ denotes the function specified in (2.6) in Section 2.3.

**Theorem 1.1.** Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}$ and $a_1, \ldots, a_N \in \mathbb{C}\setminus\{0\}$. Then, with the notation above, the following three statements are equivalent:

(a) $T$ is compact on $A^p_{\alpha}(D)$;

(b) \( \lim_{|z| \to 1} \frac{\|T\lambda_z^s\|_{A^p_{\alpha}}}{\|\lambda_z^s\|_{A^p_{\alpha}}} = 0 \) for all $s > \frac{\alpha+2}{p}$;

(c) \( \lim_{|z| \to 1} Q(z) = 0 \).

As a consequence of this result, we see that compactness of linear combination is independent of parameters $\alpha$ and $p$, as expected. We also note that Moorhouse’s characterization (1.3) of compact differences is a direct consequence of the equivalence of (a) and (c).

As an application of Theorem 1.1, we consider the class of linear combinations satisfying the coefficient non-cancellation condition

$$(\text{CNC}) \quad s_\sigma^j \neq 0 \quad \text{for} \quad j = 1, \ldots, N - 1 \quad \text{and} \quad \sigma \in \mathcal{P}_N.$$  

For this class of linear combinations, we obtain the next characterization, which has been known for the special case $N = 2$; see [4, Theorem 4.6].
Theorem 1.2. Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi_1, \varphi_2, \ldots, \varphi_N \in S$ and assume that $a_1, \ldots, a_N \in \mathbb{C} \setminus \{0\}$ satisfy (CNC). Also, assume that at least one of the operators $T_1, \ldots, T_N$ is not compact on $A^p_\alpha(D)$. Then

$$T$$ is compact on $A^p_\alpha(D)$

if and only if both of the following two conditions are fulfilled:

(a) $\sum_{j=1}^N a_j = 0$;

(b) $T_{j,k}$ is compact for each $j, k \in \Lambda_N$.

Our next result is a Carleson measure characterization. To state it we introduce more notation. Given $\sigma \in \mathcal{P}_N$, we put

$$G_\sigma := \left\{ z \in D : Q_\sigma(z) = \min_{\tau \in \mathcal{P}_N} Q_\tau(z) \right\}.$$  

(1.7)

Let $\alpha > -1$ and $0 < p < \infty$. Associated with $\sigma \in \mathcal{P}_N$ are the measures $\mu_{\sigma,j} = \mu^p_{\sigma,j}$ and $\nu_{\sigma,j} = \nu^p_{\sigma,j}$ defined by

$$
\mu_{\sigma,j}(E) := \int_{\varphi_{\sigma,j}^{-1}(E) \cap G_\sigma} M^p_{\sigma,j,\sigma,j+1} \, dA_\alpha + \int_{\varphi_{\sigma,j+1}^{-1}(E) \cap G_\sigma} M^p_{\sigma,j+1,\sigma,j+1} \, dA_\alpha,
$$

$$
\nu_{\sigma,j}(E) := \int_{\varphi_{\sigma,j}^{-1}(E) \cap G_\sigma} \rho^p_{\sigma,j,\sigma,j+1} \, dA_\alpha + \int_{\varphi_{\sigma,j+1}^{-1}(E) \cap G_\sigma} \rho^p_{\sigma,j+1,\sigma,j+1} \, dA_\alpha
$$

for $1 \leq j < N$ and

$$
\mu_{\sigma,N}(E) = \nu_{\sigma,N}(E) := \int_{\varphi_{\sigma,N}^{-1}(E) \cap G_\sigma} \rho^p_{\sigma,N} \, dA_\alpha
$$

for Borel sets $E \subset D$. Finally, we put

(1.8)

$$
\mu_{\alpha,p} := \sum_{\sigma \in \mathcal{P}_N} \mu^\sigma_{\alpha,p} \text{ where } \mu^\sigma_{\alpha,p} := \sum_{j=1}^N |s_j^\sigma|^p \mu_{\sigma,j}
$$

and

$$
\nu_{\alpha,p} := \sum_{\sigma \in \mathcal{P}_N} \nu^\sigma_{\alpha,p} \text{ where } \nu^\sigma_{\alpha,p} := \sum_{j=1}^N |s_j^\sigma|^p \nu_{\sigma,j}.
$$

For the rest of the paper we will freely use the notation introduced so far without any further reference.

Our last result is the Carleson measure characterization. When $N = 2$ and $a_1 + a_2 = 0$, it is easily seen that the set $G_\sigma$ is simply the whole $D$ for each $\sigma \in \mathcal{P}_2$. Thus, for differences of composition operators, the equivalence of (a) and (c) below is contained in [22] and [13]; see also [5] for the half-plane analogue.

Theorem 1.3. Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi_1, \varphi_2, \ldots, \varphi_N \in S$ and $a_1, \ldots, a_N \in \mathbb{C} \setminus \{0\}$. Then the following three statements are equivalent:

(a) $T$ is compact on $A^p_\alpha(D)$;

(b) $\mu_{\alpha,p}$ is a compact $\alpha$-Carleson measure on $D$;

(c) $\nu_{\alpha,p}$ is a compact $\alpha$-Carleson measure on $D$. 

We emphasize that our symbol functions in Theorems 1.1 and 1.3 are completely arbi-
trary, as long as they belong to $S$.

The exposition of the paper is organized as follows. In Section 2 we recall some basic
facts to be used in later sections. Section 3 is devoted to the proof of Theorem 1.1. Section
4 is devoted to the proof of Theorem 1.3. In Section 5 we first prove Theorem 1.2 as an
application of Theorem 1.1 and observe some consequences. We then apply our results to
provide new simple proofs for some known results about linear combinations of at most
four composition operators.

**Constants.** Throughout the paper we use the letter $C$ to denote various positive constants
which may change at each occurrence. Variables indicating the dependency of $C$ will be
often specified in a parenthesis. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative
quantities $X$ and $Y$ to mean $X \leq CY$ for some inessential constant $C > 0$. Similarly, we
use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

### 2. Preliminaries

In this section we collect some basic facts to be used in later sections. One may find
details in standard references such as [7] and [28].

#### 2.1. Compact Operator. It seems better to clarify the notion of compact operators, since
the spaces under consideration are not Banach spaces when $0 < p < 1$. Suppose $X$ and
$Y$ are topological vector spaces whose topologies are induced by complete metrics. A
continuous linear operator $S : X \to Y$ is said to be compact if the image of every bounded
sequence in $X$ has a subsequence that converges in $Y$.

We have the following convenient compactness criterion for a linear combination of
composition operators acting on the weighted Bergman spaces.

**Lemma 2.1.** Let $\alpha > -1$ and $0 < p < \infty$. Let $S$ be a linear combination of composition
operators. Then $S$ is compact on $A^p_\alpha(D)$ if and only if $Sf_n \to 0$ in $A^p_\alpha(D)$ for any bounded
sequence $\{f_n\}$ in $A^p_\alpha(D)$ such that $f_n \to 0$ uniformly on compact subsets of $D$.

A proof can be found in [7, Proposition 3.11] for a single composition operator and it
can be easily modified for a linear combination.

#### 2.2. Pseudohyperbolic Distance. The well-known pseudohyperbolic distance between
$z, w \in D$ is given by

$$\rho(z, w) := \frac{|z - w|}{1 - \overline{z}w}.$$  

By a straightforward calculation we have

$$1 - \rho^2(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2}$$  

for $z, w \in D$. The pseudohyperbolic disk with center $z \in D$ and radius $r \in (0, 1)$ is
defined by

$$E_r(z) := \{w \in D : \rho(z, w) < r\}.$$  

It turns out that $E_r(z)$ is a Euclidean disk with

$$\text{(center)} = \frac{(1 - r^2)}{1 - |z|^2r^2}z \quad \text{and} \quad \text{(radius)} = \frac{(1 - |z|^2)r}{1 - |z|^2r^2}.$$
This implies
\begin{equation}
1 - \rho(z, w) \leq \frac{1 - |z|}{1 - |w|} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}
\end{equation}
for \(z, w \in D\). Also is well known that, given \(0 < r < 1\) and \(\alpha > -1\), we have
\begin{equation}
A^2_\alpha[E_r(z)] \approx (1 - |z|)^{\alpha + 2}, \quad z \in D;
\end{equation}
constants suppressed in this estimate depend only on \(\alpha\) and \(r\).

Given \(0 < r < 1\) and \(\alpha > -1\), we will use the submean value type inequality
\begin{equation}
|f(z)|^p \leq \frac{C}{(1 - |z|)^{\alpha + 2}} \int_{E_r(z)} |f|^p dA_\alpha, \quad z \in D
\end{equation}
for functions \(f \in H(D), 0 < p < \infty\) and for some constant \(C = C(\alpha, r) > 0\).
All the details for the statements above can be found in [28, Chapter 4].

2.3. Test Function. Note from (2.5) with \(p = 2\) that each point evaluation is a continuous linear functional on the Hilbert space \(A^2_\alpha(D)\). Thus, to each \(z \in D\) corresponds a unique reproducing kernel whose explicit formula is known as
\begin{equation}
\lambda_z(w) := \frac{1}{1 - zw}, \quad w \in D
\end{equation}
for \(z \in D\).

Powers of the functions in (2.6) will be the source of test functions in conjunction with Lemma 2.1. The norms of such kernel-type functions are well known. Namely, when \(sp > \alpha + 2\), we have
\begin{equation}
\|\lambda_z^s\|_{A^p_\alpha} \approx \frac{1}{(1 - |z|)^{s + \frac{2s}{p}}}, \quad a \in D;
\end{equation}
constants suppressed in this estimate are independent of \(z\); see, for example, [28, Lemma 3.10]. Thus
\begin{equation}
\frac{\lambda_z^s}{\|\lambda_z^s\|_{A^p_\alpha}} \to 0 \quad \text{uniformly on compact subsets of } D
\end{equation}
as \(|z| \to 1\).

2.4. Carleson Measure. Let \(\alpha > -1\) and \(\mu\) be a finite positive Borel measure on \(D\). For \(0 < r < 1\) and \(0 < p < \infty\), it is well known that
the embedding \(A^p_\alpha(D) \subset L^p(d\mu)\) is bounded
\begin{equation}
\iff \sup_{z \in D} \mu[E_r(z)] A_\alpha[E_r(z)] < \infty
\end{equation}
and
\begin{equation}
\text{the embedding } A^p_\alpha(D) \subset L^p(d\mu) \text{ is compact}
\end{equation}
\begin{equation}
\iff \lim_{|z| \to 1} A_\alpha[E_r(z)] = 0.
\end{equation}
We say that \(\mu\) is an \(\alpha\)-Carleson measure if (2.9) holds. Also, we say that \(\mu\) is a compact \(\alpha\)-Carleson measure if (2.10) holds. Note that the notion of (compact) \(\alpha\)-Carleson measures is independent of the parameters \(p\) and \(r\). Given \(\alpha > -1\), it is also well known that
\begin{equation}
\|i_{\mu, p}\|^p \approx \|\mu\|_{\alpha} := \sup_{z \in D} \mu[E(z)] A_\alpha[E(z)] \quad \text{where } E(z) := E_{1/2}(z)
\end{equation}
for \(\alpha\)-Carleson measures \(\mu\); constants suppressed above are independent of \(\mu\) and \(p\). Here, \(\|i_{\mu,p}\|\) denotes the operator norm of the embedding \(i_{\mu,p} : A^p_\alpha(D) \subset L^p(d\mu)\); see [28, Section 7.2].

The connection between composition operators and Carleson measures comes from the change-of-variable formula (see [9, p. 163])

\[
\int_D (g \circ \varphi) \, dA_\alpha = \int_D g \, d(A_\alpha \circ \varphi^{-1})
\]

valid for \(\varphi \in \mathcal{S}\) and positive Borel functions \(g\) on \(D\). Here, \(A_\alpha \circ \varphi^{-1}\) denotes the pullback measure defined by \((A_\alpha \circ \varphi^{-1})(E) = A_\alpha[\varphi^{-1}(E)]\) for Borel sets \(E \subset D\). Since each composition operator is bounded on \(A^p_\alpha(D)\), it is immediate from (2.12) that \(A_\alpha \circ \varphi^{-1}\) is an \(\alpha\)-Carleson measure for each \(\varphi \in \mathcal{S}\). Also is well known via (2.12) that \(C_\varphi\) is compact on \(A^p_\alpha(D)\) if and only if \(A_\alpha \circ \varphi^{-1}\) is a compact \(\alpha\)-Carleson measure.

### 2.5. Angular Derivative

We recall the well-known notion of the angular derivative. Let \(T := \partial D\) be the unit circle. A map \(\varphi \in \mathcal{S}\) is said to have a finite angular derivative at \(\zeta \in T\), denoted by \(\varphi'(\zeta) \in \mathbb{C}\), if \(\varphi\) has nontangential limit \(\varphi(\zeta) \in T\) at \(\zeta\) such that

\[
\angle \lim_{z \to \zeta} \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta} = \varphi'(\zeta)
\]

where \(\angle \lim\) stands for the nontangential limit. As is well known by the Julia-Carathéodory Theorem (see [7, Theorem 2.44]), \(\varphi'(\zeta)\) exists if and only if

\[
\liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty.
\]

In this case, the left-hand side of the above is equal to \(|\varphi'(\zeta)|\) and, moreover,

\[
\angle \lim_{z \to \zeta} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = \frac{1}{|\varphi'(\zeta)|}.
\]

In particular, we have \(|\varphi'(\zeta)| \geq \frac{1 - |\varphi(0)|}{1 + |\varphi(0)|} > 0\) by the Schwarz-Pick lemma. We put

\[
F(\varphi) := \left\{ \zeta \in T : \limsup_{z \to \zeta} R_\varphi(z) > 0 \right\};
\]

this is the “angular derivative set” of \(\varphi\) consisting of all boundary points at which \(\varphi\) has finite angular derivatives. Note that (1.1) is equivalent to the condition \(F(\varphi) = \emptyset\).

### 3. Julia-Carathéodory Type Characterization

This section is devoted to the proof of Theorem 1.1. We will complete the proof of Theorem 1.1 by establishing the implications

\[(a) \implies (b) \implies (c) \implies (a).\]

Note that the implication \((a) \implies (b)\) is clear by Lemma 2.1 and (2.8).

Before proceeding, we fix some notation to be used throughout the section. We put

\[
T_J := \sum_{j \in J} a_j T_j
\]
for $J \subset \Lambda_N$. Given $J \subset \Lambda_N$ with $\ell \in J$, note from (1.4) that the operator $T_J$ can be written in the form

$$T_J = \sum_{j,k \in J} c_{j,k} T_{j,k} + \left( \sum_{j \in J} a_j \right) T_\ell$$

(3.1)

for some coefficients $c_{j,k}$ depending on $\ell$ and $a_j$’s with $j \in J$.

We now proceed to the proof of the implication $(b) \implies (c)$. We need the following estimate, which is the disk version of [13, Lemma 2.2]; one may also refer to [22, Lemma 3.1].

**Lemma 3.1.** Let $0 < p < \infty$, $\alpha > -1$ and $0 < r_1 < r_2 < 1$. Then there is a constant $C = C(\alpha, p, r_1, r_2) > 0$ such that

$$|f(a) - f(b)|^p \leq C \rho^p(a,b) \int_{E_{r_2}(a)} |f|^p dA_\alpha$$

(3.2)

for $f \in H(D)$ and $a, b \in D$ with $b \in E_{r_1}(a)$.

**Proof of (b) $\implies$ (c).** Assume (b). We suppose that (c) fails and will derive a contradiction. Since (c) fails, there exists a sequence $\{z_n\} \subset D$ such that $|z_n| \to 1$ and

$$\inf_n Q(z_n) > 0.$$

Since functions $Q_\sigma$ are all bounded on $D$, it follows that

$$\inf_n Q_\sigma(z_n) > 0 \text{ for each } \sigma \in \mathcal{P}_N.$$

Furthermore, since functions $R_j$ and $\rho_{j,k}$ are all bounded, we may assume that the sequences

$$\{R_j(z_n)\} \text{ and } \{\rho_{j,k}(z_n)\}$$

both converge as $n \to \infty$ for any $j, k \in \Lambda_N$. Put

$$F := \left\{ j \in \Lambda_N : \lim_{n \to \infty} R_j(z_n) = 0 \right\}$$

(3.4)

and let

$$F' := \Lambda_N \setminus F$$

for short. Passing to a subsequence if necessary, we may further assume

$$\inf_n R_j(z_n) > 0$$

(3.5)

for each $j \in F'$.

In conjunction with the set $F'$, we pause to prove the following claim.

**Claim.** There are pairwise disjoint sets of indices $J_1, \ldots, J_L$ and a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, with the following properties:

(i) $F' = \bigcup_{m=1}^L J_m$;

(ii) To each $m = 1, \ldots, L$ corresponds $k_m \in J_m$ such that

$$|\varphi_j(z_n)| \leq |\varphi_{k_m}(z_n)| \text{ for all } n \text{ and } j \in \bigcup_{\ell=m}^L J_\ell;$$

(iii) $\lim_{n \to \infty} \rho_{j,k}(z_n) = 0$ for all $j, k \in J_m$ for each $m = 1, \ldots, L$. 
(iv) \( \sum_{j \in J_m} a_j = 0 \) for each \( m = 1, \ldots, L \).

**Proof of Claim.** Assume \( F' \neq \emptyset \) to avoid triviality. Put \( J_0 := F \) for convenience. Passing to a subsequence if necessary, we may choose an index \( k_1 \notin J_0 \) such that
\[
|\varphi_j(z_n)| \leq |\varphi_{k_1}(z_n)| \quad \text{for all } n \text{ and } j \notin J_0.
\]
Using such \( k_1 \), we set
\[
J_1 := \left\{ j \in F' : \lim_{n \to \infty} \rho_{k_1,j}(z_n) = 0 \right\}.
\]
Now, let \( \nu \geq 2 \) and suppose that we have chosen pairwise disjoint sets \( J_m \) and points \( k_m \in J_m \) for \( m = 1, \ldots, \nu - 1 \) which satisfy
\[
|\varphi_j(z_n)| \leq |\varphi_{k_m}(z_n)| \quad \text{for all } n \text{ and } j \notin \bigcup_{\ell=0}^{m-1} J_\ell \tag{3.6}
\]
and
\[
J_m = \left\{ j \in F' : \lim_{n \to \infty} \rho_{k_m,j}(z_n) = 0 \right\} \tag{3.7}
\]
for \( m = 1, \ldots, \nu - 1 \). If \( F' = \bigcup_{m=1}^{\nu-1} J_m \), then we stop. Otherwise, passing to a further subsequence if necessary, we choose an index
\[
k_\nu \notin \bigcup_{m=0}^{\nu-1} J_m \tag{3.8}
\]
such that
\[
|\varphi_j(z_n)| \leq |\varphi_{k_\nu}(z_n)| \quad \text{for all } n \text{ and } j \notin \bigcup_{m=0}^{\nu-1} J_m.
\]
Using this \( k_\nu \), we set
\[
J_\nu := \left\{ j \in F' : \lim_{n \to \infty} \rho_{k_\nu,j}(z_n) = 0 \right\}.
\]
Note from (3.8) and (3.7) that \( J_\nu \) and \( J_m \) are disjoint for each \( m = 1, \ldots, \nu - 1 \). This process stops after finitely many steps by pairwise disjointness of the sets. Thus, (i) holds for some \( L \). Note that (ii) holds by (3.6) and (i). Also, since \( k_m \in J_m \) for each \( m \), (iii) is clear by (3.7).

For (iv), we first consider the case \( m = 1 \). For \( s > 0 \), put
\[
f_a = f_{a,s} := (1 - |a|^2)^s \lambda_n^{\frac{a+2}{2}+s}
\]
for \( a \in \mathbf{D} \). Put
\[
w_n := \varphi_{k_1}(z_n)
\]
for each \( n \). Note \( 1 - |w_n|^2 \approx 1 - |z_n|^2 \) by (3.5). So, \( |w_n| \to 1 \). It follows from (2.7) and (b) that
\[
\lim_{n \to \infty} \|Tf_{w_n}\|_{A_0^\mu} = 0.
\]
We thus obtain
\[
\lim_{n \to \infty} (1 - |w_n|^2)^{\frac{a+2}{2}} |Tf_{w_n}(z_n)| = 0 \tag{3.9}
\]
by (2.5).

Note from (iii) and (2.3) that
\[
1 - |w_n|^2 \approx 1 - |\varphi_j(z_n)|^2, \quad j \in J_1
\]
for all $n$. It follows from Lemma 3.1 and (2.4) that
\begin{equation}
(1 - |w_n|^2)\omega^{2}||T_{j,k}f_{w_n}(z_n)||^p \lesssim |f_{j,k}(z_n)| \to 0, \quad j, k \in J_1
\end{equation}
as $n \to \infty$. Meanwhile, note
\begin{equation}
(1 - |w_n|^2)\omega^{2}||T_{j,k}f_{w_n}(z_n)||^p = \left( \frac{1 - |w_n|^2}{(1 - \varphi_j(z_n)w_n)} \right)^{\omega^{2} + \alpha^{2}}, \quad j \in \Lambda_N
\end{equation}
for all $n$. In particular,
\begin{equation}
(1 - |w_n|^2)\omega^{2}||T_{k,l}f_{w_n}(z_n)||^p = 1
\end{equation}
for all $n$. Thus, applying (3.1) (with $J = J_1$ and $\ell = k_1$), we obtain
\begin{equation}
\lim_{n \to \infty} (1 - |w_n|^2)^{\frac{\omega^{2} + \alpha^{2}}{2}} ||T_{j,l}f_{w_n}(z_n)||^p = \sum_{j \in J_1} a_j.
\end{equation}
Since
\begin{equation}
(1 - |w_n|^2)\omega^{2}||T_{j,k}f_{w_n}(z_n)||^p \lesssim R_{j}^{\omega^{2} + \alpha^{2}}(z_n) \to 0, \quad j \in F
\end{equation}
by (3.11) and definition of the set $F$, we also note
\begin{equation}
\lim_{n \to \infty} (1 - |w_n|^2)\omega^{2}||T_{F}f_{w_n}(z_n)||^p = 0.
\end{equation}

We now consider operators $T_{j,k}$, $1 < k \leq L$. Let $j \in J_k$, $1 < k \leq L$. We have $|\varphi_j(z_n)| \leq |w_n|$ by (ii) and thus
\begin{equation}
\left( \frac{1 - |w_n|^2}{(1 - \varphi_j(z_n)w_n)} \right)^2 \leq \frac{(1 - |w_n|^2)(1 - |\varphi_j(z_n)|^2)}{(1 - \varphi_j(z_n)w_n)^2} = 1 - \rho_{1,j}^2(z_n)
\end{equation}
for all $n$. It follows from (3.11)
\begin{equation}
(1 - |w_n|^2)\omega^{2}||T_{j,k}f_{w_n}(z_n)||^p \leq \left(1 - \rho_{1,j}^2(z_n)\right)^{\frac{\omega^{2} + \alpha^{2}}{2}}
\end{equation}
for all $n$. Thus, setting
\begin{equation}
\eta_1 := \min \left[ \lim_{n \to \infty} \rho_{k_1,j}(z_n) \right] > 0
\end{equation}
where the minimum is taken over all $j \in \cup_{1 < k \leq L} J_k$, we obtain
\begin{equation}
\limsup_{n \to \infty} (1 - |w_n|^2)\omega^{2} \sum_{1 < k \leq L} ||T_{j,k}f_{w_n}(z_n)||^p \leq C_1 (1 - \eta_1^2)^{\frac{\omega^{2} + \alpha^{2}}{2}}
\end{equation}
for some constant $C_1 = C_1(p, N, a_1, \ldots, a_N) > 0$. Now, since we have by (i)
\begin{equation}
T_{j_1} = T - T_F - \sum_{1 < m \leq L} T_{j_m},
\end{equation}
we deduce from (3.9), (3.12), (3.13) and (3.14) that
\begin{equation}
\left| \sum_{j \in J_1} a_j \right| \leq C_1 (1 - \eta_1^2)^{\frac{\omega^{2} + \alpha^{2}}{2}}.
\end{equation}
Recall that $s > 0$ is arbitrary. So, taking the limit $s \to \infty$, we conclude
\begin{equation}
\sum_{j \in J_1} a_j = 0,
\end{equation}
as required. This completes the proof for the case $m = 1$.  

COMPACT LINEAR COMBINATION 11
We now proceed by induction on \( m \). So, let \( 2 \leq m \leq L \) and assume
\[
\sum_{j \in J_1} a_j = \cdots = \sum_{j \in J_{m-1}} a_j = 0. \tag{3.15}
\]
We will show
\[
\sum_{j \in J_m} a_j = 0. \tag{3.16}
\]
This time our test functions are \( f_{\xi_n} \) where \( \xi_n = \varphi_{k_m}(z_n) \) for each \( n \). Using (3.1), (3.5) and the induction hypothesis (3.15), we have as in the proof of (3.10)
\[
\lim_{n \to \infty} (1 - |\xi_n|^2)^{1/2} \sum_{1 \leq k < m} |T_{J_k} f_{\xi_n}(z_n)|^p = 0.
\]
Also, as in the case of \( m = 1 \), we obtain
\[
\lim_{n \to \infty} (1 - |\xi_n|^2)^{1/2} |T_{F} f_{\xi_n}(z_n)|^p = 0
\]
and
\[
\lim_{n \to \infty} (1 - |\xi_n|^2)^{1/2} |T_{J_m} f_{\xi_n}(z_n)| = \left| \sum_{j \in J_m} a_j \right|.
\]
Setting
\[
\eta_m := \min \left[ \lim_{n \to \infty} \rho_{k_m,j}(z_n) \right] > 0
\]
where the minimum is taken over all \( j \in \bigcup_{m < k \leq L} J_k \), we also have
\[
\limsup_{n \to \infty} (1 - |\xi_n|^2)^{1/2} \sum_{m < k \leq L} |T_{J_k} f_{\xi_n}(z_n)|^p \leq C_1 (1 - \eta_m^2)^{p+2/2}.
\]
Now, as in the case of \( m = 1 \), we conclude (3.16), which completes the induction. So, (iv) holds. The proof of the claim is complete. \( \square \)

Having proved the claim above, we now continue the proof of the implication (b) \( \Rightarrow \) (c). We may assume that the sets \( J_1, \ldots, J_L, F \) are all nonempty; otherwise the proof is simpler. Let \( n_m := \# J_m \), the number of elements in \( J_m \), and put
\[
j_m := (j_{m,1}, \ldots, j_{m,n_m}), \quad m = 1, \ldots, L
\]
where \( j_{m,1}, \ldots, j_{m,n_m} \) are the distinct elements of \( J_m \). Also, let \( d := \# F \) and put
\[
f := (j_1, \ldots, j_d)
\]
where \( j_1, \ldots, j_d \) are the distinct elements of \( F \). Now, consider \( \tau \in \mathcal{P}_N \) given by
\[
\tau := (j_1, \ldots, j_L, \hat{f}).
\]
Using this \( \tau \), we may rephrase Claim (iv) as
\[
\sum_{j=1}^{n_1 + \cdots + n_m} a_{\tau_j} = 0 \tag{3.17}
\]
for \( m = 1, \ldots, L \). This yields
\[
\sum_{j=1}^{N-d} |s_j^T| M_{\tau_j, \tau_{j+1}} = \sum_{m=1}^{L} \sum_{j,k \in J_m} c_{j,k}^{(m)} M_{j,k}
\]
where \(c_{j,k}^{(m)}\) are nonnegative coefficients depending on \(\{a_\ell : \ell \in J_m\}\) for each \(m\). On the other hand, note

\[
\sum_{j=N-d+1}^{N-1} |s^T_j| M_{\tau_j,\tau_{j+1}} + |s^T_N| R_{\tau_N} \leq \sum_{j \in F} c_j R_j
\]

where \(c_j\) are nonnegative coefficients depending on \(\{a_{j_1}, \ldots, a_{j_d}\}\). It follows that

\[
Q_\tau = \sum_{j=1}^{N-1} |s^T_j| M_{\tau_j,\tau_{j+1}} + |s^T_N| R_{\tau_N}
\]

\[
\leq \sum_{m=1}^L \sum_{j,k \in J_m} c_{j,k}^{(m)} M_{j,k} + \sum_{j \in F} c_j R_j.
\]

Note that the first term of the above tends to 0 along the sequence \(\{z_n\}\) by Claim (iii). The second term also tends to 0 along the sequence \(\{z_n\}\) by definition (3.4) of the set \(F\). Consequently, we obtain

\[
\lim_{n \to \infty} Q_\tau(z_n) = 0,
\]

which is a contradiction to (3.3). The proof is complete. \(\square\)

We now proceed to the proof of the implication (c) \(\implies\) (a). We recall a couple of known estimates. First, we recall the following estimate which is implicit in the proof of [19, Lemma 1] or [4, Lemma 4.3].

\begin{lemma}
Let \(\alpha > -1\) and \(0 < p < \infty\). Put \(\gamma := \min\{\frac{\alpha+1}{2}, 1\}\). Let \(\varphi \in S\), \(\epsilon > 0\) and \(W : D \to [0,1]\) be a Borel function. If

\[
\sup_{z \in D} \left[ W(z) \frac{1-|z|^2}{1-|\varphi(z)|^2} \right] \leq \epsilon,
\]

then there is a constant \(C = C(\alpha) > 0\) such that

\[
\int_D |f \circ \varphi|^p W \, dA_\alpha \leq C \epsilon^\gamma \|f\|_{A^p_\alpha}^p
\]

for \(f \in A^p_\alpha(D)\).
\end{lemma}

Next, the following lemma is taken from [6, Lemma 3.3].

\begin{lemma}
Let \(\alpha > -1\) and \(0 < p < \infty\). Let \(\varphi, \psi \in S\), \(\epsilon > 0\) and \(K \subset D\) be a Borel set. If

\[
\sup_{z \in K} \left[ \frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\psi(z)|^2} \right] \rho(\varphi(z), \psi(z)) \leq \epsilon,
\]

then there is a constant \(h(\epsilon) > 0\) such that

\[
\lim_{\epsilon \to 0} h(\epsilon) = 0
\]

and

\[
\int_K |f \circ \varphi - f \circ \psi|^p \, dA_\alpha \leq h(\epsilon) \|f\|_{A^p_\alpha}^p
\]

for \(f \in A^p_\alpha(D)\).
We are now ready to prove the implication (c) \implies (a). In the proof below we use the notation
\[
\Omega_\delta(\zeta) := \left\{ z \in D : 1 - \delta < |z| < 1, \left| \zeta - \frac{z}{|z|} \right| < \delta \right\},
\]
for \( \zeta \in T \) and \( 0 < \delta < 1 \).

**Proof of (c) \implies (a).** Assume (c). Our proof relies on Lemma 2.1. So, consider an arbitrary sequence \( \{f_n\} \) in \( A^p_\alpha(D) \) such that \( \sup_n \|f_n\|_{A^p_\alpha} \leq 1 \) and \( f_n \to 0 \) uniformly on compact sets of \( D \). According to Lemma 2.1, it suffices to show that \( Tf_n \to 0 \) in \( A^p_\alpha(D) \).

Let \( \epsilon > 0 \) and put
\[
U_{\sigma,\epsilon} := \{ z \in D : Q_\sigma(z) \leq \epsilon \}
\]
for \( \sigma \in \mathcal{P}_N \). Note from (c) that to each \( \zeta \in T \) corresponds \( \delta_\zeta = \delta_\zeta(\epsilon) \in (0,1) \) such that \( \Omega_{\delta_\zeta}(\zeta) \subset \bigcup_{\sigma \in \mathcal{P}_N} U_{\sigma,\epsilon} \).

Indeed, if this does not hold, then there would be some \( \zeta_0 \in T \) with the following property:
For any \( \delta \in (0,1) \) the set \( \Omega_\delta(\zeta_0) \) contains a point \( z_0 \in D \) such that \( Q_\sigma(z_0) > \epsilon \) for all \( \sigma \in \mathcal{P}_N \) and hence \( Q(z_0) > \epsilon N! \), which is impossible by (c).

Now, by compactness of \( T \), we can find finitely many points \( \zeta_1, \ldots, \zeta_\nu \) in \( T \) such that
\[
D \setminus rD \subset \bigcup_{j=1}^\nu \Omega_{\delta_j}(\zeta_j) \subset \bigcup_{\sigma \in \mathcal{P}_N} U_{\sigma,\epsilon}
\]
where \( \delta_j := \delta_{\zeta_j} \) and \( r := \max_j (1 - \delta_j) \); the second inclusion above holds by (3.20). It follows that
\[
\int_D |Tf_n|^p \, dA_\alpha \leq \int_{rD} + \sum_{\sigma \in \mathcal{P}_N} \int_{U_{\sigma,\epsilon}} := I_n + \sum_{\sigma \in \mathcal{P}_N} II_n^{\sigma,\epsilon}
\]
for each \( n \). For the first term of the above, note that \( f_n \to 0 \) uniformly on \( \bigcup_{j=1}^N \varphi_j(rD) \), which is contained in a compact set of \( D \). It follows that
\[
I_n \to 0
\]
as \( n \to \infty \). To estimate the second term, let \( \sigma \in \mathcal{P}_N \). Recall that \( Q_\sigma \leq \epsilon \) on \( U_{\sigma,\epsilon} \). Thus, in conjunction with the representation (1.4) of \( T \), we see from Lemma 3.3 that there is a constant \( h_\sigma(\epsilon) > 0 \) such that \( \lim_{\epsilon \to 0} h_\sigma(\epsilon) = 0 \) and
\[
\sum_{j=1}^{N-1} |s_j^\sigma|^p \int_{U_{\sigma,\epsilon}} |T_{\sigma_j,\sigma_{j+1}} f_n|^p \, dA_\alpha \leq h_\sigma(\epsilon)
\]
for all \( n \). In addition, we see from Lemma 3.2 that there are constants \( C > 0 \) and \( \gamma > 0 \), independent of \( \epsilon \), such that
\[
|s_N^\sigma|^p \int_{U_{\sigma,\epsilon}} |T_{\sigma N} f_n|^p \, dA_\alpha \leq C \epsilon^\gamma
\]
for all \( n \). Accordingly, there is a constant \( C > 0 \), independent of \( \epsilon \), such that
\[
II_n^{\sigma,\epsilon} \leq C[h_\sigma(\epsilon) + \epsilon^\gamma]
\]
for all $n$. We now conclude by (3.21) and (3.22) that
\[
\limsup_{n \to \infty} \int_D |Tf_n|^p \, dA_\alpha \leq h(\epsilon)
\]
for some constant $h(\epsilon) > 0$ satisfying $h(\epsilon) \to 0$ as $\epsilon \to 0$. Finally, taking the limit $\epsilon \to 0$ in the right-hand side of the above, we conclude, as required, that $Tf_n \to 0$ in $A^p_\alpha(D)$. The proof is complete.

4. CARLESON MEASURE CHARACTERIZATION

This section is devoted to the proof of Theorem 1.3. Before proceeding, we observe a couple of basic properties of the joint pullback measures defined in (1.8). Let $\alpha > -1$ and $0 < p < \infty$. Let $\sigma \in \mathcal{P}_N$. We note from definitions of $Q_\sigma$ and $\mu^\sigma_{\alpha,p}$

\[(4.1) \quad \mu^\sigma_{\alpha,p}(E) \leq 2 \sum_{j=1}^{N-1} \int_{\psi_j^{-1}(E) \cap G_\sigma} Q^\sigma_\alpha \, dA_\alpha
\]
for Borel sets $E \subset D$. We also note via the standard argument that led to (2.12) the change-of-variable formula

\[(4.2) \quad \int_D g \, d\mu^\sigma_{\alpha,p} = \sum_{j=1}^{N-1} |s^\sigma_j|^p \int_{G_\sigma} [g \circ \varphi_{\sigma_j} + g \circ \varphi_{\sigma_{j+1}}] \, M^p_{\varphi_{\sigma_j}, \varphi_{\sigma_{j+1}}} \, dA_\alpha
\]
valid for positive Borel functions $g$ on $D$.

**Lemma 4.1.** Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi, \psi \in \mathcal{S}$ and $W : D \to [0,1]$ be a Borel function. Let $\mu$ and $\nu$ be the measures defined by

\[
\mu(E) : = \int_{\varphi^{-1}(E)} (R_\varphi + R_\psi)^p W \, dA_\alpha + \int_{\psi^{-1}(E)} (R_\varphi + R_\psi)^p W \, dA_\alpha
\]

and

\[
\nu(E) : = \int_{\varphi^{-1}(E)} W \, dA_\alpha + \int_{\psi^{-1}(E)} W \, dA_\alpha
\]
for Borel sets $E \subset D$. If $\mu$ is a compact $\alpha$-Carleson measure on $D$, then so is $\nu$.

**Proof.** Let $\epsilon \in (0,1)$ and put
\[
K_\epsilon := \{z \in D : R_\varphi(z) + R_\psi(z) \leq \epsilon\}.
\]
Let $E \subset D$ be a Borel set. Note
\[
\int_{\varphi^{-1}(E)} W \, dA_\alpha = \int_{\varphi^{-1}(E) \cap K_\epsilon} + \int_{\varphi^{-1}(E) \setminus K_\epsilon} W \, dA_\alpha
\]
\[
\leq \int_{\varphi^{-1}(E)} \chi_{K_\epsilon} W \, dA_\alpha + \frac{1}{\epsilon^p} \int_{\varphi^{-1}(E)} (R_\varphi + R_\psi)^p W \, dA_\alpha
\]
where $\chi_{K_\epsilon}$ is the characteristic function of the set $K_\epsilon$. The same estimates holds for $\psi$. It follows that
\[
\nu \leq \nu_\epsilon + \frac{1}{\epsilon^p} \mu
\]
where \( \nu \) is the restriction of \( \nu \) to \( K \). Accordingly, using the notation introduced in (2.11), we obtain

\[
\frac{\nu [E(z)]}{A_\alpha [E(z)]} \leq \nu_\epsilon |_\alpha + \frac{1}{e^p} \frac{\mu [E(z)]}{A_\alpha [E(z)]}
\]

for all \( z \in D \). Now, assuming that \( \mu \) is a compact \( \alpha \)-Carleson measure on \( D \) and letting \( |z| \to 1 \), we obtain

\[
\limsup_{|z| \to 1} \frac{\nu [E(z)]}{A_\alpha [E(z)]} \leq \| \nu_\epsilon \|_\alpha
\]

for each \( \epsilon > 0 \). Note from Lemma 3.2 and (2.11) that \( \| \nu_\epsilon \|_\alpha \to 0 \) as \( \epsilon \to 0 \). Thus, taking the limit \( \epsilon \to 0 \), we conclude the lemma. The proof is complete. \( \square \)

We will complete the proof of Theorem 1.3 by establishing the equivalences

(a) \( \iff \) (b) and (b) \( \iff \) (c).

Note that the implication (b) \( \implies \) (c) holds by Lemma 4.1. The implication (c) \( \implies \) (b) is clear, because functions \( R_j \) are all bounded. It remains to establish the equivalence (a) \( \iff \) (b).

In what follows we put

\[ E(z) := E_{\frac{1}{2}}(z), \quad z \in D \]

for short. Also, we put

\[ D^r := D \setminus rD \]

for \( 0 < r < 1 \).

**Proof of (a) \( \implies \) (b).** Assume that \( T \) is compact on \( A_\alpha^p(D) \). Let \( \sigma \in \mathcal{P}_N \). Note from definition (1.7) of the set \( G_\sigma \)

\[
[Q_\sigma(w)]^{N!} \leq Q(w), \quad w \in G_\sigma.
\]

Thus, given \( \epsilon > 0 \), we see from Theorem 1.1 and (4.3) that there is \( r \in (0, 1) \) such that

\[
Q_\sigma(w) \leq \left[ Q(w) \right]^{\frac{1}{N!}} < \epsilon, \quad w \in G_\sigma \cap D^r.
\]

Choose \( r_0 \in (0, 1) \) such that

\[
\varphi_j(rD) \subset r_0D, \quad \text{or equivalently,} \quad \varphi_j^{-1}(D^{r_0}) \subset D^r
\]

for each \( j \).

Note from (2.2) that there is \( r_1 \in (0, 1) \) such that \( E(z) \subset D^{r_0} \) for \( z \in D^{r_1} \). It follows from the observations in the preceding paragraph that

\[
Q_\sigma < \epsilon \quad \text{on} \quad \varphi_j^{-1}[E(z)] \cap G_\sigma, \quad z \in D^{r_1}
\]

for each \( j \). This, together with (4.1), yields

\[
\mu_{\alpha,p}^\sigma[E(z)] \leq 2e^p \sum_{j=1}^N (A_\alpha \circ \varphi_j^{-1})(E(z)), \quad z \in D^{r_1}
\]

so that

\[
\sup_{z \in D^{r_1}} \frac{\mu_{\alpha,p}^\sigma[E(z)]}{A_\alpha [E(z)]} \leq 2e^p \sum_{j=1}^N \| A_\alpha \circ \varphi_j^{-1} \|_\alpha.
\]
Note \( \| A_\alpha \circ \varphi_j^{-1} \|_\alpha < \infty \) for each \( j \); see the remark after (2.12). It follows from the above that
\[
\lim_{|z| \to 1} \mu_{\alpha,p}^\sigma [E(z)] \circ A_\alpha = 0,
\]
or said differently, that \( \mu_{\alpha,p}^\sigma \) is a compact \( \alpha \)-Carleson measure. Since \( \sigma \in \mathcal{P}_N \) is arbitrary, we finally conclude that \( \mu_{\alpha,p} \) is a compact \( \alpha \)-Carleson measure. The proof is complete. \( \square \)

**Proof of (b) \implies (a).** Assume that \( \mu_{\alpha,p} \) is a compact \( \alpha \)-Carleson measure. As in the proof of Theorem 1.1, our proof relies on Lemma 2.1. So, consider an arbitrary sequence \( \{ f_n \} \) in \( A_{\alpha}^p(D) \) such that \( \sup_n \| f_n \|_{A_{\alpha}^p} \leq 1 \) and \( f_n \to 0 \) uniformly on compact sets of \( D \). We need to show that \( T f_n \to 0 \) in \( A_{\alpha}^p(D) \).

Fix \( r \in (0, 1) \). Since \( f_n \to 0 \) uniformly on \( rD \) and \( \bigcup_{\sigma \in \mathcal{P}_N} G_{\sigma} = D \), we have
\[
\limsup_{n \to \infty} \int_D |T f_n|^p \, dA_\alpha = \limsup_{n \to \infty} \int_{D r} |T f_n|^p \, dA_\alpha 
\leq \limsup_{n \to \infty} \sum_{\sigma \in \mathcal{P}_N} \int_{G_{\sigma}^r} |T f_n|^p \, dA_\alpha \tag{4.4}
\]
where \( G_{\sigma}^r := G_{\sigma} \cap D r \).

Fix \( \sigma \in \mathcal{P}_N \). Let \( \epsilon > 0 \). With the representation (1.4) in mind, put
\[
I_n^{\sigma, r, j} := \int_{G_{\sigma}^r \cap \{ M_{\sigma_j, \sigma_{j+1}} \geq \epsilon \}} |T_{\sigma_j, \sigma_{j+1}} f_n|^p \, dA_\alpha 
\]
and
\[
II_n^{\sigma, r, j} := \int_{G_{\sigma}^r \cap \{ M_{\sigma_j, \sigma_{j+1}} < \epsilon \}} |T_{\sigma_j, \sigma_{j+1}} f_n|^p \, dA_\alpha \tag{4.5}
\]
for each \( 1 \leq j < N \) and \( n \). Similarly, put
\[
I_n^{\sigma, r, N} := \int_{G_{\sigma}^r \cap \{ R_{\sigma, N} \geq \epsilon \}} |T_{\sigma(N)} f_n|^p \, dA_\alpha 
\]
and
\[
II_n^{\sigma, r, N} := \int_{G_{\sigma}^r \cap \{ R_{\sigma, N} < \epsilon \}} |T_{\sigma N} f_n|^p \, dA_\alpha \tag{4.7}
\]
for each \( n \). Using these notation and the representation (1.4) of \( T \), we have
\[
\int_{G_{\sigma}^r} |T f_n|^p \, dA_\alpha \leq \sum_{j=1}^{N} |s_j|^p (I_n^{\sigma, r, j} + II_n^{\sigma, r, j}) \tag{4.9}
\]
for each \( n \).

For the integrals in (4.5) and (4.7), we have
\[
I_n^{\sigma, r, j} \leq \frac{1}{\epsilon^p} \int_{G_{\sigma}^r} |T_{\sigma_j, \sigma_{j+1}} f_n|^p M_{\sigma_j, \sigma_{j+1}}^p \, dA_\alpha, \quad 1 \leq j < N
\]
and
\[
I_n^{\sigma, r, N} \leq \frac{1}{\epsilon^p} \int_{G_{\sigma}^r} |T_{\sigma N} f_n|^p R_{\sigma N}^p \, dA_\alpha
\]
for all $n$. Now, since
\[ |T_{\sigma_j,\sigma_{j+1}}f_n|^p \lesssim |T_{\sigma_j}f_n|^p + |T_{\sigma_{j+1}}f_n|^p \]
it follows from (4.2) that
\[ \sum_{j=1}^{N} |s_j^\sigma P_n^{\sigma,\epsilon,j} \lesssim \epsilon^p \int_{D} |f_n|^p d\mu_\sigma \]
for all $n$; the constant suppressed above depends only on $p$. Note that $\mu_\sigma$ is a compact $\alpha$-Carleson measure, because $\mu_{\alpha,p}$ is by assumption. Also, recall that $\sup_n \|f_n\|_{A_\alpha^p} \leq 1$ and $f_n \to 0$ uniformly on compact sets of $D$. Thus the above estimate implies
\[ \lim_{n \to \infty} \sum_{j=1}^{N} |s_j^\sigma P_n^{\sigma,\epsilon,j} = 0 \]  
(4.10)

Meanwhile, for the integrals in (4.6) we have by Lemma 3.3 a constant $h_\sigma(\epsilon) > 0$ such that
\[ \lim_{\epsilon \to 0} h_\sigma(\epsilon) = 0 \]
and
\[ \sum_{j=1}^{N-1} |s_j^\sigma P I_n^{\sigma,\epsilon,j} \leq h_\sigma(\epsilon) \]  
(4.11)
for all $n$. For the integrals in (4.8) we have by Lemma 3.2 constants $C > 0$ and $\gamma > 0$, independent of $\epsilon$, such that
\[ |s_N^\sigma|^p |P I_n^{\sigma,\epsilon,N} \leq C \epsilon^\gamma \]  
(4.12)
for all $n$. Now, we see from (4.9), (4.10), (4.11) and (4.12)
\[ \limsup_{n \to \infty} \int_{G_\alpha} |Tf_n|^p dA \leq h_\sigma(\epsilon) + C \epsilon^\gamma. \]
So, taking the limit $\epsilon \to 0$, we obtain
\[ \lim_{n \to \infty} \int_{G_\alpha} |Tf_n|^p dA \to 0 \]
and thus conclude by (4.4) that $Tf_n \to 0$ in $A_\alpha^p(D)$, as required. The proof is complete. \qed

5. Applications

In this section we first prove Theorem 1.2 as an application of Theorem 1.1. Also, applying our results, we will recover some known results about compactness of linear combinations of composition operators.

We begin with the simple lemma below, which will be used repeatedly in our proofs later. Recall that $R_\varphi$ and $M_{\varphi,\psi}$ are the functions defined in (1.1) and (1.3).

**Lemma 5.1.** Let $\varphi, \psi \in S$. Let $\{z_n\}$ be a sequence in $D$ such that
\[ \inf_n R_\varphi(z_n) > 0 \]  
and
\[ \lim_{n \to \infty} R_\psi(z_n) = 0. \]
Then
\[ \lim_{n \to \infty} \rho(\varphi(z_n), \psi(z_n)) = 1 \]
and
\[ \limsup_{n \to \infty} M_{\varphi,\psi}(z_n) > 0. \]
Proof. By (2.1) we have
\[ 1 - \rho^2(\varphi, \psi) \leq \frac{1}{1 - |\psi|^2} - \frac{1}{1 - |\varphi|^2}. \]
Note that the right-hand side of the above tends to 0 along the sequence \(\{z_n\}\) by (5.1). So, the first part of the lemma holds. The second part is also clear by (5.1) and the first part. \(\square\)

We now prove Theorem 1.2.

Proof of Theorem 1.2. The sufficiency is obvious by (1.4). For the necessity, suppose that \(T\) is compact on \(A^p_\alpha(D)\). We first prove (a). Due to the compactness of \(T\), we have by Theorem 1.1
\[ \lim_{|z| \to 1} \prod_{\sigma \in \mathcal{P}_N} Q_\sigma(z) = 0. \]
Consider an arbitrary sequence \(\{z_n\}\) such that \(|z_n| \to 1\). We see from the above that there exists \(\tau \in \mathcal{P}_N\) and a subsequence \(\{z_{n_\ell}\}\) such that
\[ \lim_{\ell \to \infty} Q_\tau(z_{n_\ell}) = 0. \]
This, together with (CNC), yields
\[ \lim_{\ell \to \infty} M_{\tau_j, \tau_{j+1}}(z_{n_\ell}) = 0 \]
for \(1 \leq j < N\).

We now assume that (a) fails and will derive a contradiction to complete the proof of (a). Since (a) fails, we have in addition to (5.2)
\[ \lim_{\ell \to \infty} R_{\tau_N}(z_{n_\ell}) = 0. \]
This, together with Lemma 5.1 and (5.2) with \(j = N - 1\), yields \(\lim_{\ell \to \infty} R_{\tau_{N-1}}(z_{n_\ell}) = 0\).

Repeating the same argument, we have
\[ \lim_{\ell \to \infty} R_j(z_{n_\ell}) = 0 \]
for each \(j = 1, \ldots, N\). This subsequential property implies
\[ \lim_{|z| \to 1} R_j(z) = 0, \]
for each \(j = 1, \ldots, N\). Thus operators \(T_1, \ldots, T_N\) are all compact on \(A^p_\alpha(D)\) by the characterization (1.1) due to Shapiro and Taylor, which is impossible by assumption that at least one of the operators is not compact on \(A^p_\alpha(D)\). This completes the proof of (a).

We now turn to the proof of (b). Having proved (a), we note that (b) is trivial for \(N = 2\). So, we assume \(N \geq 3\). We introduce some temporary notation. For \(k \in \Lambda_N\) with \(k \neq 1\), let \(B_k (\neq \emptyset)\) be the set of all \(\sigma \in \mathcal{P}_N\) such that
\[ \{\sigma_j, \sigma_{j+1}\} = \{1, k\} \quad \text{for some} \quad j \neq N \]
and put
\[ Q_{B_k} := \prod_{\sigma \in B_k} Q_\sigma. \]
Note that each $\sigma \in \mathcal{P}_N$ belongs to at least one and at most two of the sets $B_2, \ldots, B_N$. Since functions $Q_\sigma$ are all bounded, it follows that

$$\prod_{k=2}^N Q_{B_k} \leq CQ$$

for some constant $C > 0$.

To prove (b), it suffices to show that $T_{1,j}$ is compact on $A^p_\alpha(D)$ for each $j = 2, \ldots, N$. Suppose not. For simplicity we may suppose that $T_{1,2}$ is not compact on $A^p_\alpha(D)$. By the characterization (1.3) due to Moorhouse, there is a sequence $\{z_n\}$ in $D$ such that $|z_n| \to 1$ and

$$\inf_n \{(R_1(z_n) + R_2(z_n))|\rho_{1,2}(z_n)\} > 0.$$  

In addition, since $R_1$ and $R_2$ are bounded on $D$, we may also assume

$$\inf_n \rho_{1,2}(z_n) > 0.$$  

Note also from (5.4) that $Q_{B_2}$ stays away from 0 along the sequence $\{z_n\}$. It follows from (5.3) and Theorem 1.1 that

$$\lim_{n \to \infty} \prod_{k=3}^N Q_{B_k}(z_n) = 0.$$  

This implies that one of the factors tends to 0 along a subsequence, still denoted by $\{z_n\}$. So, we may assume $Q_{B_3}(z_n) \to 0$. By the same argument we may further assume

$$Q_{\eta}(z_n) \to 0$$

for some $\eta \in B_3$. Since

$$Q_{\eta} = \sum_{j=1}^{N-1} |s^\eta_j|M_{\eta_j,\eta_{j+1}}$$

by (a), it follows from (CNC) that (5.6) is equivalent to

$$\lim_{n \to \infty} M_{\eta_j,\eta_{j+1}}(z_n) = 0$$

for each $1 \leq j < N$.

Note from (5.4) that (after passing to a subsequence if necessary)

$$\inf_n R_{j_0}(z_n) > 0$$

for some $j_0 \in \{1, 2\}$. It follows from Lemma 5.1 and (5.7) that (after passing to a subsequence if necessary)

$$\inf_n R_{j_1}(z_n) > 0,$$

when $j_1$ is adjacent to $j_0$, i.e., when $\{j_0, j_1\} = \{\eta_j, \eta_{j+1}\}$ for some $1 \leq j < N$. Starting with the above, we see by the same argument that (after passing to a subsequence if necessary)

$$\inf_n R_{j_2}(z_n) > 0$$

when $j_2$ is adjacent to $j_1$. Repeating the same arguments, we can find a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that

$$\inf_n R_j(z_n) > 0$$
for each \( j \in \Lambda_N \). This, together with (5.7), implies
\[
\rho_{i,j,j+1}(z_n) \to 0
\]
for each \( j \in \Lambda_N \). It follows that \( \rho_{1,2}(z_n) \to 0 \), which is a contradiction to (5.5). The proof is complete. \( \square \)

We now notice an easy special case of Theorem 1.2, which might be of independent interests.

**Corollary 5.2.** Let \( \alpha > -1 \) and \( 0 < p < \infty \). Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \in S \). Assume \( c_j > 0 \) for each \( j = 2, \ldots, N \). Then
\[
\sum_{j=2}^{N} c_j T_{1,j} \text{ is compact on } A^p_\alpha(D)
\]
if and only if
\[
T_{j,k} \text{ is compact on } A^p_\alpha(D) \text{ for each } j, k \in \Lambda_N.
\]

**Proof.** The sufficiency being trivial, we only need to prove the necessity. Before proceeding, we note
\[
\sum_{j=2}^{N} c_j T_{1,j} = \sum_{j=1}^{N} a_j T_j
\]
where \( a_1 = \sum_{j=2}^{N} c_j \) and \( a_j = -c_j \) for \( j = 2, \ldots, N \). It is easy to see that (CNC) holds and \( \sum_{j=1}^{N} a_j = 0 \).

If all the operators \( T_1, \ldots, T_N \) are compact on \( A^p_\alpha(D) \), there is nothing to prove. If at least one of them is not compact on \( A^p_\alpha(D) \), then we see from Theorem 1.2 that \( T_{j,k} \) is compact on \( A^p_\alpha(D) \) for each \( j, k \in \Lambda_N \). The proof is complete. \( \square \)

**Example.** Applying Theorem 1.2, we can easily check whether a linear combination of composition operators, for which (CNC) is fulfilled, is compact on \( A^p_\alpha(D) \). For example:

- \( 6T_1 - T_2 - 2T_3 - 3T_4 \) is compact on \( A^p_\alpha(D) \) if and only if \( T_{j,k} \) is compact on \( A^p_\alpha(D) \) for each \( j, k = 1, 2, 3, 4 \).
- \( 4T_1 - 2T_2 - T_3 \) is compact on \( A^p_\alpha(D) \) only when \( T_1, T_2, T_3 \) are all compact on \( A^p_\alpha(D) \).
- \( T_1 - iT_2 - 2T_3 \) is compact on \( A^p_\alpha(D) \) only when \( T_1, T_2, T_3 \) are all compact on \( A^p_\alpha(D) \).

In [14] the last two of the current authors characterized compactness of 3-combinations. Here, we provide new proofs based on Theorem 1.1 in the next three corollaries. First, when (CNC) is satisfied, we have the next immediate consequence of Theorem 1.2.

**Corollary 5.3 ([14]).** Let \( \alpha > -1 \) and \( 0 < p < \infty \). Let \( \varphi_1, \varphi_2, \varphi_3 \in S \) and \( a_1, a_2, a_3 \in \mathbb{C} \setminus \{0\} \). Assume \( a_j + a_k \neq 0 \) for each \( j, k = 1, 2, 3 \). Also, assume that at least one of the operators \( T_1, T_2, T_3 \) is not compact on \( A^p_\alpha(D) \). Then
\[
a_1T_1 + a_2T_2 + a_3T_3 \text{ is compact on } A^p_\alpha(D)
\]
if and only if both of the following two conditions are fulfilled:

(i) \( a_1 + a_2 + a_3 = 0 \);
(ii) \( T_1 - T_2 \) and \( T_1 - T_3 \) are both compact on \( A^p_\alpha(D) \).
Next, when (CNC) is not satisfied, note that 3-combination reduces (after re-indexing) to a nontrivial combination of $T_1 - T_2$ and $T_3$. In conjunction with this observation we have the next corollary.

**Corollary 5.4** ([14]). Let $\alpha > -1$ and $0 < p < \infty$. Let $a \in \mathbb{C} \setminus \{0\}$. Let $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}$. Suppose that none of $T_1, T_2, T_3$ is compact on $\mathcal{A}_a^p(\mathcal{D})$. If $T_1 - T_2 + aT_3$ is compact on $\mathcal{A}_a^p(\mathcal{D})$, then $a = 1$ or $a = -1$.

**Proof.** We first compute the function $Q$ associated with the operator under consideration. Note

$$
T := (T_1 - T_2) + aT_3 \\
= (T_1 - T_3) + (1 + a)(T_3 - T_2) + aT_2 \\
= -(T_2 - T_1) + aT_3 \\
= -(T_2 - T_3) + (-1 + a)(T_3 - T_1) + aT_1 \\
= a(T_3 - T_1) + (a + 1)(T_1 - T_2) + aT_2 \\
= a(T_3 - T_2) + (a - 1)(T_2 - T_1) + aT_1
$$

Thus, setting

$$S_1 := M_{1,2} + |a|R_3 \\
S_2 := M_{1,3} + |a + 1|M_{3,2} + |a|R_2 \\
S_3 := M_{2,1} + |a|R_3 \\
S_4 := M_{2,3} + |a - 1|M_{3,1} + |a|R_1 \\
S_5 := |a|M_{3,1} + |a + 1|M_{1,2} + |a|R_3 \\
S_6 := |a|M_{3,2} + |a - 1|M_{2,1} + |a|R_1$$

we have

$$Q = S_1S_2S_3S_4S_5S_6.$$ 

Now, assume $T$ is compact on $\mathcal{A}_a^p(\mathcal{D})$. Note that $T_1 - T_2$ is not compact. Thus there is a sequence $\{z_n\}$ in $\mathcal{D}$ with $|z_n| \to 1$ such that

$$\inf_n M_{1,2}(z_n) > 0. \tag{5.8}$$

Assume $a \neq \pm 1$. Then $S_4S_3S_2S_1$ stays away from 0 along the sequence $\{z_n\}$. Accordingly, we have $\lim_{n \to \infty} S_2(z_n)S_4(z_n) = 0$ by Theorem 1.1. Thus, we may assume either $\lim_{n \to \infty} S_2(z_n) = 0$ or $\lim_{n \to \infty} S_4(z_n) = 0$ after passing a subsequence if necessary. Suppose $\lim_{n \to \infty} S_2(z_n) = 0$, i.e.,

(i) $\lim_{n \to \infty} R_2(z_n) = 0$, 
(ii) $\lim_{n \to \infty} M_{1,3}(z_n) = 0$, 
(iii) $\lim_{n \to \infty} M_{2,3}(z_n) = 0$.

By (i), (iii) and Lemma 5.1 we have $\lim_{n \to \infty} R_3(z_n) = 0$. This, together with (ii) and Lemma 5.1 again, in turn yields

$$\lim_{n \to \infty} R_1(z_n) = 0,$$

which is impossible by (5.8) and (i). We thus conclude $a = 1$ or $a = -1$, as desired. The proof is complete. \qed
Finally, when (CNC) is not satisfied, note from Corollary 5.4 that 3-combination reduces (after re-indexing) to a constant multiple of \( T_1 - T_2 - T_3 \). In conjunction with this observation we have the next corollary. Recall that \( F(\varphi) \) denotes the angular derivative set of \( \varphi \) defined in (2.14).

**Corollary 5.5 ([14]).** Let \( \alpha > -1 \) and \( 0 < p < \infty \). Let \( \varphi_1, \varphi_2, \varphi_3 \in \mathcal{S} \). Suppose that none of \( T_1, T_2, T_3 \) is compact on \( A^p_D(\mathbb{D}) \). Then

\[
T_1 - T_2 - T_3 \text{ is compact on } A^p_D(\mathbb{D})
\]

if and only if both of the following two conditions are fulfilled:

1. \( F(\varphi_1) \) is the disjoint union of \( F(\varphi_2) \) and \( F(\varphi_3) \);
2. \( \lim_{z \to \zeta} M_{1,j}(z) = 0 \) for each \( \zeta \in F(\varphi_j) \) and \( j = 2, 3 \).

**Proof.** By Theorem 1.1 it suffices to prove

\[
\lim_{|z| \to 1} Q(z) = 0 \iff (a) \text{ and } (b) \text{ hold}.
\]

In conjunction with this assertion, we note that the function \( Q \) associated with \( T_1 - T_2 - T_3 \) is given by

\[
Q = (M_{1,2} + R_3)^2(M_{1,3} + R_2)^2 \times (M_{2,3} + 2M_{1,3} + R_1)(M_{2,3} + 2M_{1,2} + R_1)
\]

by the proof of Corollary 5.4. Using this explicit expression of \( Q \), one may easily verify that (a) and (b) imply \( Q(z) \to 0 \) as \( z \to \zeta \) for any \( \zeta \in T \). Thus the implication \( \Leftarrow \) holds.

We now prove the implication \( \Rightarrow \). So, assume

\[
\lim_{|z| \to 1} Q(z) = 0.
\]

We first prove (a). Let \( \zeta_0 \in F(\varphi_1) \). We have

\[
\inf_n R_1(z_n) > 0
\]

for some sequence \( \{ z_n \} \) with \( z_n \to \zeta_0 \). We have by (5.9) and (5.10)

\[
(M_{1,2} + R_3)(M_{1,3} + R_2) \to 0 \quad \text{along the sequence } \{ z_n \}.
\]

Now, suppose \( \zeta_0 \notin F(\varphi_2) \) so that \( R_2(z_n) \to 0 \). Note from Lemma 5.1 that

\[
\inf_n M_{1,2}(z_n) > 0
\]

after passing to a subsequence if necessary. It follows from the above that \( M_{1,3}(z_n) \to 0 \). Accordingly, we have

\[
R_3(z_n) \not\to 0
\]

by Lemma 5.1 and thus \( \zeta_0 \in F(\varphi_3) \). We therefore conclude

\[
F(\varphi_1) \subset F(\varphi_2) \cup F(\varphi_3).
\]

To prove the converse inclusion, suppose not and pick \( \zeta_1 \in F(\varphi_2) \cup F(\varphi_3) \) such that \( \zeta_1 \notin F(\varphi_1) \). We may assume \( \zeta_1 \in F(\varphi_2) \) by symmetry so that

\[
\inf_n R_2(w_n) > 0
\]

for some sequence \( \{ w_n \} \) with \( w_n \to \zeta_1 \). Meanwhile, since \( \zeta_1 \notin F(\varphi_1) \), we have

\[
R_1(w_n) \to 0
\]
and thus $\inf_n M_{1,2}(w_n) > 0$ by Lemma 5.1 after passing to a subsequence if necessary. Now we have by (5.10) and (5.9)

(i) $\lim_{n \to \infty} M_{2,3}(w_n) = 0$ and (ii) $\lim_{n \to \infty} M_{1,3}(w_n) = 0$.

Applying Lemma 5.1, we note from (5.13) that (i) implies $R_3(z_n) \not\to 0$, and thus (5.14) and (ii) implies $R_3(z_n) \to 0$. This contradiction shows that the inclusion in (5.12) can be reversed, as required.

If $\zeta_2 \in F(\varphi_2) \cap F(\varphi_3) \subset F(\varphi_1)$, then we have by (2.13)

\[ \angle \lim_{z \to \zeta_2} (R_1 R_2 R_3)(z) = \frac{1}{|\varphi_1'(\zeta_2)||\varphi_2'(\zeta_2)||\varphi_3'(\zeta_2)|} > 0, \]

which is impossible by (5.9). So, we conclude $F(\varphi_2) \cap F(\varphi_3) = \emptyset$. This completes the proof of (a).

We now prove (b). Suppose not. By symmetry we may assume that there is $\zeta_3 \in F(\varphi_2)$ such that $\limsup_{z \to \zeta_3} M_{1,2}(z) > 0$. Pick a sequence $\{\xi_n\} \subset D$ with $\xi_n \to \zeta_3$ such that

(iii) $\inf_n R_1(\xi_n) > 0$ or (iv) $\inf_n R_2(\xi_n) > 0$

after passing to a subsequence if necessary. Since $\zeta_3 \in F(\varphi_2)$, we note $\zeta_3 \notin F(\varphi_3)$ and thus $R_3(\xi_n) \to 0$.

In case of (iii), we see from (5.15) and (5.9) that $M_{1,3}(\xi_n) + R_2(\xi_n) \to 0$. So, we have $\zeta_3 \in F(\varphi_3)$ as in the proof of (5.11), which is a contradiction. In case of (iv), since $R_3(\xi_n) \to 0$, we have by Lemma 5.1

\[ \inf_n M_{2,3}(\xi_n) > 0 \]

after passing to a subsequence if necessary. So, we see from (5.15), (iv), (5.16) and (5.9) that $Q(\xi_n) \not\to 0$, which is also a contradiction to (5.10). We thus conclude (b). The proof is complete.

The current authors have recently characterized compactness of double differences of composition operators. Here, we provide a new proof below, which is much simpler (thanks to Theorem 1.1) than the original one.

**Corollary 5.6 ([6]).** Let $\alpha > -1$ and $0 < p < \infty$. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in S$. Then

\[ (T_1 - T_2) - (T_3 - T_4) \text{ is compact on } A^p_\alpha(D) \]

if and only if

\[ \lim_{|z| \to 1} [M_{1,2}(z) + M_{3,4}(z)][M_{1,3}(z) + M_{2,4}(z)] = 0. \]
Proof. We first identify the function $Q$ associated with the operator under consideration. Note

\[
T : = (T_1 - T_2) - (T_3 - T_4) \\
= (T_1 - T_3) - (T_2 - T_4) \\
= (T_1 - T_4) + 2(T_4 - T_2) + (T_2 - T_3) \\
= - (T_2 - T_3) - 2(T_3 - T_1) - (T_1 - T_4) \\
= - (T_2 - T_3) - 2(T_3 - T_4) - (T_4 - T_1) \\
= - (T_3 - T_2) - 2(T_2 - T_1) - (T_1 - T_4) \\
= - (T_3 - T_2) - 2(T_2 - T_4) - (T_4 - T_1) \\
= (T_4 - T_1) + 2(T_1 - T_2) + (T_2 - T_3) \\
= (T_1 - T_3) + 2(T_1 - T_3) + (T_3 - T_2).
\]

Associated with these representations, put

\[
S_1 := M_{1,2} + M_{3,4}, \quad S_2 := M_{1,3} + M_{2,4} \\
S_3 := M_{1,4} + 2M_{4,2} + M_{2,3}, \quad S_4 := M_{1,4} + 2M_{4,3} + M_{3,2} \\
S_5 := M_{2,3} + 2M_{3,1} + M_{1,4}, \quad S_6 := M_{2,3} + 2M_{3,4} + M_{4,1} \\
S_7 := M_{3,2} + 2M_{2,1} + M_{1,4}, \quad S_8 := M_{3,2} + 2M_{2,4} + M_{4,1} \\
S_9 := M_{4,1} + 2M_{1,2} + M_{2,3}, \quad S_{10} := M_{4,1} + 2M_{1,3} + M_{3,2}.
\]

It is routine to check

\[
Q = (S_1 S_2)^8 (S_3 S_4 \cdots S_9 S_{10}).
\]

Hence, to complete the proof, it is sufficient to prove the equivalence

\[
\lim_{|z| \to 1} Q(z) = 0 \iff \lim_{|z| \to 1} S_1(z) S_2(z) = 0
\]

by Theorem 1.1. The implication $\iff$ is clear.

For the implication $\implies$, we suppose that it fails and will derive a contradiction. So, assume $Q(z) \to 0$ but $S_1(z) S_2(z) \not\to 0$ as $|z| \to 1$. Then there is a sequence $\{z_n\}$ in $D$ with $|z_n| \to 1$ such that

\[
\inf_n S_1(z_n) > 0, \quad \inf_n S_2(z_n) > 0
\]

and $\lim_{n \to \infty} S_j(z_n) = 0$ for some $j \geq 3$. We consider the case $j = 3$; proofs for other cases are similar. So, we have $\lim_{n \to \infty} S_3(z_n) = 0$, or said differently,

(i) $\lim_{n \to \infty} M_{1,4}(z_n) = 0$, \\
(ii) $\lim_{n \to \infty} M_{2,3}(z_n) = 0$, \\
(iii) $\lim_{n \to \infty} M_{2,4}(z_n) = 0$.

By (5.20) and (iii) we have

\[
\inf_n M_{1,3}(z_n) > 0
\]

after passing to a subsequence if necessary. We now split the proof into two cases as follows.

\[
\inf_n M_{1,3}(z_n) > 0
\]
First, consider the case when \( R_1(z_n) \to 0 \). In this case we have by (i) and Lemma 5.1
\[
R_4(z_n) \to 0.
\]
We also have \( R_3(z_n) \not\to 0 \) by (5.21). It follows from (ii) that
\[
\lim_{n \to \infty} \rho_{2,3}(z_n) = 0,
\]
after passing to a subsequence if necessary. Therefore we have \( R_2(z_n) \not\to 0 \) by Lemma 5.1. By the same argument based on (iii) we have
\[
\lim_{n \to \infty} \rho_{2,4}(z_n) = 0
\]
after passing to a subsequence if necessary and thus \( R_4(z_n) \not\to 0 \), which is a contradiction to (5.22).

Next, consider the case when \( R_1(z_n) \not\to 0 \). In this case note from (i)-(iii) and Lemma 5.1 that \( R_j(z_n) \not\to 0 \) for \( j = 1, 2, 3, 4 \). Accordingly, we see from (i)-(iii) again that
\[
\rho_{1,4}(z_n) + \rho_{2,3}(z_n) + \rho_{2,4}(z_n) \to 0,
\]
after passing to a subsequence if necessary. This implies \( \rho_{1,3}(z_n) \to 0 \), which is also a contradiction to (5.21). The proof is complete. \( \square \)

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