TOEPLITZ PRODUCTS WITH PLURIHARMONIC SYMBOLS
ON THE HARDY SPACE OVER THE BALL

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ABSTRACT. On the Hardy space over the unit ball in $\mathbb{C}^n$, we consider operators which have the form of a finite sum of products of several Toeplitz operators. We study characterizing problems of when such an operator is compact or of finite rank. Some of our results show higher dimensional phenomena.

1. INTRODUCTION

For a fixed positive integer $n$, let $B$ be the open unit ball of the complex $n$-space $\mathbb{C}^n$ and let $S$ be the unit sphere, the boundary of $B$. In case $n = 1$ we use more customary notation $D$ and $T$ in place of $B$ and $S$, respectively.

Given $0 < p < \infty$, we denote by $H^p(B)$ the Hardy space on $B$ consisting of all holomorphic functions $f$ on $B$ such that

$$\|f\|_{H^p} := \sup_{0 < r < 1} \left\{ \int_S |f(r\zeta)|^p \, d\sigma(\zeta) \right\}^{1/p} < \infty$$

where $\sigma$ denotes the surface area measure on $S$ normalized to have total mass 1. We also denote by $H^\infty(B)$ the space of all bounded holomorphic functions on $B$.

As is well known, the space $H^p(B)$, $1 \leq p < \infty$, is a Banach space and isometrically identified with a closed subspace of $L^p(S) = L^p(S, \sigma)$ via the (admissible) boundary functions; see Section 2 for the notion of boundary function. In particular, $H^2(B)$ is a Hilbert space. It is also well known that the Hilbert space orthogonal projection $Q : L^2(S) \to H^2(B)$ is realized by the integral operator

$$(1.1) \quad Qu(z) = \int_S \frac{u(\zeta)}{(1 - z \cdot \overline{\zeta})^n} \, d\sigma(\zeta), \quad z \in B$$

for $u \in L^2(S)$. Here and elsewhere, $z \cdot \overline{w} = \sum_{j=1}^n z_j \overline{w}_j$ denotes the Hermitian inner product of $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ on $\mathbb{C}^n$.

For $u \in L^\infty(S)$, the Toeplitz operator $T_u$ with symbol $u$ is an operator on $H^2(B)$ defined by

$$T_u f = Q(u f)$$

for $f \in H^2(B)$. Note that we are using the same notation $f \in H^2(B)$ and its boundary function $f \in L^2(S)$. Clearly, $T_u$ is linear and bounded on $H^2(B)$.

Throughout the paper we are mainly concerned with pluriharmonic symbols. Recall that a twice continuously differentiable function $\psi$ on $B$ is said to be pluriharmonic if the
one-variable function $\lambda \mapsto \psi(a + b)$, defined for $\lambda \in \mathbb{C}$ such that $a + \lambda b \in \mathbb{B}$, is harmonic for each $a \in \mathbb{B}$ and $b \in \mathbb{C}^m$. As is well known, a function $\psi$ on $\mathbb{B}$ is pluriharmonic if and only if $\psi = f + \overline{g}$ for some functions $f$ and $g$ holomorphic on $\mathbb{B}$; see [28, Theorem 4.4.9]. Also, note that such $f$ and $g$ are uniquely determined up to additive constants. So, we may say that $f$ and $g$ are the holomorphic part and the co-holomorphic part, respectively, of $\psi$. It is also well known that the holomorphic part of a bounded pluriharmonic function belongs to $BMOA(\mathbb{B})$ (see [34] for definition) and hence to all $H^p(\mathbb{B})$; see [24, Proposition 4] for example.

We denote by $ph^\infty(\mathbb{B})$ the space of all bounded pluriharmonic functions on $\mathbb{B}$. Also, we denote by $ph^\infty(\mathbb{S})$ the space of all functions $u \in L^\infty(\mathbb{S})$ such that $Pu \in ph^\infty(\mathbb{B})$ where $Pu$ denotes the Poisson-Szegö integral of $u$; see Section 2 for precise definition.

As is well known, the spaces $ph^\infty(\mathbb{B})$ and $ph^\infty(\mathbb{S})$ are isometrically isomorphic via boundary functions and the Poisson-Szegö integral transform. Thus we can freely identify a function in $ph^\infty(\mathbb{B})$ with its boundary function in $ph^\infty(\mathbb{S})$, and vice versa. So, we will often use the same letter $\psi \in ph^\infty(\mathbb{B})$ and its boundary function $\psi \in ph^\infty(\mathbb{S})$. In particular, when we write $T_u$ with $u \in ph^\infty(\mathbb{B})$, the symbol of $T_u$ should be understood to be the boundary function of $u$.

In this paper we consider operators which is a sum of several Toeplitz products. Namely, we consider operators of the form

$$L = \sum_{i=1}^{N} \prod_{j=1}^{M} T_{u_{ij}}$$

where each $u_{ij} \in L^\infty(\mathbb{S})$. Unlike the one-variable case, we believe that this type of operators over higher dimensional balls has not been studied before except for few special cases as in [32] or [33] where Zheng characterized zero (semi)commutators with pluriharmonic symbols. We also refer to [11] and [12] for earlier works on some general aspects of Toeplitz operators in several variables. The methods, which are employed here for the several-variable theory and have emerged from our earlier works ([8], [10]) on Bergman spaces, are quite different from those in the literature for the one-variable theory and still apply to the one-variable case. Note $ph^\infty(\mathbb{T}) = L^\infty(\mathbb{T})$. So, we hope that our method would shed new light on the one-variable theory of Toeplitz operators with bounded symbols.

For an operator as in (1.2), we first provide a new proof for a known necessary condition for compactness asserting that an operator of the form (1.2) is a compact perturbation of a Toeplitz operator only when it is a compact perturbation of the Toeplitz operator with symbol $\sum_{i=1}^{N} \prod_{j=1}^{M} u_{ij}$, which can be easily derived from some general facts from [11] and [12]; see the comment after Proposition 3.1. Here, our proof is based on an observation (Theorem 3.2) on admissible boundary behavior of the so-called Poisson-Szego transform of products of Toeplitz operators; the main reason for employing this new approach is that such an idea may extend to the Bergman space setting as in Section 5. We then show for $n \geq 2$ that if a Toeplitz product, with pluriharmonic symbols continuous on some nonempty relatively open set in $\mathbb{S}$, is compact, then one of symbols is zero; see Theorem 3.6. This reveals a higher dimensional phenomenon whose one-variable analogue is easily seen to be false; see the comment after Theorem 3.6. In the course of the proof we establish a uniqueness property of pluriharmonic functions valid only on higher dimensional balls; see Theorem 3.4.

Next, we give a characterization for which an operator of the form (1.2) has finite rank; Proposition 4.1. In particular, for operators which have the form of a finite sum of products
of two Toeplitz operators with one of the symbols pluriharmonic and the other arbitrary, we give a more concrete characterization in Theorem 4.2. Also, some immediate consequences that might be of some independent interest are given; see Section 4. In Section 5 we include some corresponding results on the Hardy space over the polydisk and the Bergman space over the unit ball.

2. PREREQUISITES

In this section we collect some basic notions and related facts that are needed in the rest of the paper.

2.1. M-harmonic function. For $a \in B$ the standard automorphism $\varphi_a$ of $B$ is given by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - z \cdot \overline{a}}, \quad z \in B$$

where $P_a$ denotes the orthogonal projection onto the subspace generated by $a$ and $Q_a$ denotes the orthogonal projection onto its orthogonal complement. In fact $\varphi_a$ is an involution of $B$ that exchanges the origin and the point $a$ and, in addition, $\varphi_a : S \to S$ is one-to-one and onto. Given $u \in C^2(B)$, the so-called invariant Laplacian $\tilde{\Delta} u$ of $u$ is defined by

$$(\tilde{\Delta} u)(z) = \Delta(u \circ \varphi_z)(0), \quad z \in B$$

where $\Delta$ denotes the (ordinary) Laplacian. Functions annihilated by $\tilde{\Delta}$ are called $M$-harmonic functions. We remark in passing that term “invariant” comes from the automorphism invariance $\tilde{\Delta}(u \circ \varphi_z) = (\tilde{\Delta} u) \circ \varphi_z$ for all $z \in B$ and that in terms of ordinary differential operators the invariant Laplacian $\tilde{\Delta}$ is given by

$$\tilde{\Delta} u(z) = 4(1 - |z|^2) \sum_{i,j=1}^n (\delta_{ij} - z_i \overline{z_j}) \frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}(z)$$

where $\delta_{ij}$ is the Kronecker delta. We refer to [28, Chapter 4] or [30, Chapter 3] for details and related facts.

2.2. Poisson-Szegö integral. We denote by $P(z, \zeta)$ the Poisson-Szegö kernel defined by

$$P(z, \zeta) = \left( \frac{1 - |z|^2}{|1 - z \cdot \overline{\zeta}|^2} \right)^n$$

for $z \in B$ and $\zeta \in S$. This is the kernel for the Dirichlet problem associated with the invariant Laplacian; see [28, Theorem 3.3.4] or [30, Theorem 5.5]. More generally, the Poisson-Szegö integral of $u \in L^1(S)$ given by

$$Pu(z) = \int_S u(\zeta) P(z, \zeta) d\sigma(\zeta)$$

is an $M$-harmonic function on $B$ whose boundary function recovers $u$ a.e. on $S$ (see Subsection 2.5).

The Poisson-Szegö integral transform is automorphism invariant in the sense that $P(u \circ \varphi_a) = (Pu) \circ \varphi_a$ for all $a \in B$; see [28, Section 3]. Evaluating at the origin, we see that the kernel $P(a, \zeta)$ plays the role of the Jacobian for integrals of composite functions with $\varphi_a(\zeta)$ on $S$. More explicitly, we have

(2.1) $$\int_S u(\varphi_a(\zeta)) d\sigma(\zeta) = \int_S u(\zeta) P(a, \zeta) d\sigma(\zeta)$$
for \( a \in \mathbf{B} \) and Borel functions \( u \) on \( \mathbb{S} \) whenever the integrals make sense. So, for a function \( u \in L^\infty(\mathbb{S}) \) which is continuous on some nonempty relatively open set \( W \subset \mathbb{S} \), it is not hard to see via the dominated convergence theorem that \( Pu \) continuously extends to \( \mathbf{B} \cup W \) and \( Pu = u \) on \( W \).

2.3. Poisson-Szegő transform. Given \( z \in \mathbf{B} \), let \( K_z \) be the Cauchy-Szegő kernel given by

\[
K_z(w) = \frac{1}{(1 - w \cdot \overline{z})^n}, \quad w \in \mathbf{B}
\]

and denote by \( k_z = K_z/\|K_z\|_{H^2} \) the normalized kernel. Note \(|k_z(\zeta)|^2 = P(z, \zeta)\). Thus the Poisson-Szegő integral \( Pu \) with \( u \in L^\infty(\mathbb{S}) \) can be written as

\[
Pu(z) = \langle uk_z, k_z \rangle = \langle T_u k_z, k_z \rangle
\]

where, and in what follows, \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( L^2(\mathbb{S}) \). So, it is natural to extend the notion of Poisson-Szegő integrals to operators. Namely, given a bounded linear operator \( L \) on \( H^2(\mathbf{B}) \), we define its Poisson-Szegő transform \( P[L] \) on \( \mathbf{B} \) by

\[
P[L](z) = \langle Lk_z, k_z \rangle, \quad z \in \mathbf{B}.
\]

Note that the function \((z, w) \mapsto \langle LK_w, K_z \rangle\) is holomorphic in \( z \) and \( \overline{w} \). It is well known ([6, Proposition II.4.7]) that if a function holomorphic in \( z \) and \( \overline{w} \) and vanishes for \( z = w \), then it must vanish identically. It follows that the Poisson-Szegő transform is one-to-one.

2.4. Conjugate operator. Given \( z \in \mathbf{B} \), let \( U_z \) denote the weighted composition operator on \( H^2(\mathbf{B}) \) defined by

\[
U_zf = (f \circ \varphi_z)k_z.
\]

It is immediate from the change-of-variable formula (2.1) that \( U_z \) is an isometry. It is also straightforward to check that \( U_z \) is invertible and \( U_z^{-1} = U_z \). Now, being an invertible linear isometry, \( U_z \) is unitary.

Given a bounded linear operator \( L \) on \( H^2(\mathbf{B}) \), we define its “conjugate” operator \( L_z \) at \( z \in \mathbf{B} \) by

\[
L_z := U_zLU_z.
\]

The reason why we consider these operators lies in the representation

\[
P[L](z) = \langle L_z^1, 1 \rangle
\]

which is often useful in dealing with the Poisson-Szegő transform. Note that, since \( U_z \) is unitary, we have

\[
(L_1 \cdots L_N)_z = (L_1^z) \cdots (L_N^z)
\]

for bounded linear operators \( L_1, \ldots, L_N \) on \( H^2(\mathbf{B}) \). Also, for Toeplitz operators, it is routine to see

\[
(T_u)_z = T_{u \circ \varphi_z}
\]

for \( u \in L^\infty(\mathbb{S}) \).
2.5. Admissible limit. Given $\alpha > 1$ and $\zeta \in S$, we denote by $\Gamma_\alpha(\zeta)$ the admissible approach region with vertex $\zeta$ and aperture $\alpha > 1$, i.e.,

$$
\Gamma_\alpha(\zeta) = \left\{ z \in B : |1 - z \cdot \zeta| < \frac{\alpha}{2}(1 - |z|^2) \right\}.
$$

We say that a function $\psi : B \to C$ has an admissible limit, denoted by $\psi^*(\zeta) \in C$, at $\zeta \in S$ if

$$
\lim_{z \to \zeta, z \in \Gamma_\alpha(\zeta)} \psi(z) = \psi^*(\zeta)
$$

for every $\alpha > 1$. It is well known that a function in $L^1(S)$ is recovered by the admissible limit of its Poisson-Szeg"o integral. More explicitly, if $u \in L^1(S)$, then $(Pu)^*(\zeta) = u(\zeta)$ at every Lebesgue point (with respect to the nonisotropic balls) $\zeta$ of $u$; see [28, Theorem 5.4.8]. So, we will often use the same letter to denote a function in $L^1(S)$ and its Poisson-Szeg"o integral. This should cause no confusion from the context.

3. Compact Operators

In this section we observe a higher dimensional phenomenon for compactness of Toeplitz products. We first recall the following necessary condition for compactness of more general operators of the form (1.2). Much more is known on the disk; see Remark (2) at the end of this section.

**Proposition 3.1.** Let $L$ be as in (1.2). If $L$ is compact, then $\sum_{i=1}^N \prod_{j=1}^M u_{ij} = 0$ a.e. on $S$.

We were not able to locate an explicit reference for the above proposition in the literature, but it easily follows from two general facts as follows. Let $T$ be the Toeplitz algebra, i.e., the closed subalgebra in the algebra of all bounded linear operators on $H^2(B)$ generated by Toeplitz operators. Also, let $T$ be the Toeplitz semicommutator ideal in $T$, i.e., the closed ideal in $T$ generated by semicommutators $T_{uv} - TuTv$. It is known ([12, Theorem 2.2]) that the natural projection $u \mapsto Tu + T$ from $L^\infty(S)$ onto $T/I$ is an isometric $*$-isomorphism. It is also known ([11]) that $T$ contains all compact operators on $H^2(B)$. As a consequence we have Proposition 3.1.

Here, with extension to the Bergman space setting in mind, we provide another proof of Proposition 3.1, depending on the following admissible boundary behavior of the Poisson-Szeg"o transform.

**Theorem 3.2.** Let $L = T_{u_1} \cdots T_{u_N}$ where $u_1, \ldots, u_N \in L^\infty(S)$. Then the following statements hold.

(a) If $u_{ij}$ is continuous at $\zeta \in S$ for each $j$, then $P[L]$ continuously extends to $B \cup \{\zeta\}$ and $P[L](\zeta) = (u_1 \cdots u_N)(\zeta)$.

(b) If $\zeta \in S$ is a Lebesgue point of $u_{ij}$ for each $j$, then $P[L]$ has an admissible limit at $\zeta$ and $P[L]^*(\zeta) = (u_1 \cdots u_N)(\zeta)$.

(c) $P[L]$ has admissible limits at almost all points of $S$ and $P[L]^* = u_1 \cdots u_N$ a.e. on $S$.

**Proof.** Before proceeding we note by (2.2) and (2.3)

$$
L_z = T_{u_1 \circ \varphi_z} \cdots T_{u_N \circ \varphi_z},
$$

where, and in the rest of the proof, $z \in B$ is an arbitrary point.
First, we show (a). Assume that each $u_j$ is continuous at $\zeta \in S$. By (3.1) it is sufficient to show
\begin{equation}
T_{u_1 \circ \phi_z} \cdots T_{u_N \circ \phi_z} 1 \to (u_1 \cdots u_N)(\zeta) \quad \text{in} \quad L^2(S).
\end{equation}
Since $u_1$ is continuous at $\zeta$ and $\varphi_z(\eta) \to \zeta$ as $z \to \zeta$ for each $\eta \in S$, we have
\[
\lim_{z \to \zeta} \int_S \| (u_1 \circ \phi_z)(\eta) - u_1(\zeta) \|^2 d\sigma(\eta) = 0
\]
by continuity and the dominated convergence theorem. Thus $u_1 \circ \phi_z \to u_1(\zeta)$ in $L^2(S)$ as $z \to \zeta$. Thus, applying the projection $Q$, we see that (3.2) holds for $N = 1$.

We now proceed by induction on $N$. Assume $N \geq 2$ and suppose that the theorem holds for $N-1$. Set
\[
g_z = T_{u_2 \circ \phi_z} \cdots T_{u_N \circ \phi_z} 1 \quad \text{and} \quad g = (u_2 \cdots u_N)(\zeta)
\]
for short. Using the fact $\|T_{u_1 \circ \phi_z}\| \leq \|u_1\|_{L^\infty}$, we have
\[
\|T_{u_1 \circ \phi_z} g_z - u_1(\zeta)g\|_{H^2} \leq \|u_1\|_{L^\infty}\|g_z - g\|_{H^2} + \|T_{u_1 \circ \phi_z} g - u_1(\zeta)g\|_{H^2}.
\]
As $z \to \zeta$, the first term above converges to 0 by induction hypothesis and the second term converges to 0 by what we’ve proved above for $N = 1$. This completes the induction and hence the proof of (3.2).

Next, we show (b). Assume that $\zeta \in S$ is a Lebesgue point of $u_j$ for each $j$. Since $\zeta$ is a Lebesgue point of $u_1 \in L^\infty(S)$, we see that $\zeta$ is also a Lebesgue point of $|u_1 - u_1(\zeta)|^2$. It follows that
\[
P[|u_1 - u_1(\zeta)|^2](z) \to 0 \quad z \to \zeta \quad \text{admissibly,}
\]
or said differently,
\[
\int_S \| (u_1 \circ \phi_z)(\eta) - u_1(\zeta) \|^2 d\sigma(\eta) \to 0 \quad \text{as} \quad z \to \zeta \quad \text{admissibly.}
\]
Now, the rest of the proof is the same as above.

Finally, (c) follows from (b), because almost all points of $S$ are Lebesgue points of $u_j$ for each $j$. The proof is complete. \qed

Now, Proposition 3.1 is an easy consequence of Theorem 3.2.

Another proof of Proposition 3.1. Suppose that $L$ is compact. Note that $k_z$ weakly converges to 0 as $|z| \to 1$. Since a compact operator maps weakly convergent sequences into norm convergent ones, we have
\[
P[L](z) = \langle Lk_z, k_z \rangle \to 0 \quad \text{as} \quad |z| \to 1
\]
so that $P[L] \in C_0(B)$. On the other hand, we have by Theorem 3.2(b)
\[
P[L] = \sum_{i=1}^N \prod_{j=1}^M u_{ij}
\]
a.e. on $S$. This completes the proof. \qed

We now recall a uniqueness property for harmonic functions on $B$. Given a function $f \in C^1(B)$, we denote by $\mathcal{R}f$ the radial derivative defined by
\[
\mathcal{R}f = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j} + \sum_{j=1}^n \bar{z}_j \frac{\partial f}{\partial \bar{z}_j}.
\]
The following is a special case of [7, Proposition 4.1].
Lemma 3.3. Let \( \psi \) be a harmonic function on \( B \) and assume \( \psi \in C(B \cup W) \) for some nonempty relatively open set \( W \subset S \). If both \( \psi \) and \( R \psi \) vanish on \( W \), then \( \psi = 0 \) on \( B \).

Applying this, we obtain the following uniqueness property for pluriharmonic functions on higher dimensional balls, which certainly does not extend to the disk.

Theorem 3.4 (\( n \geq 2 \)). Let \( \psi \) be a pluriharmonic function on \( B \) and assume \( \psi \in C(B \cup W) \) for some nonempty relatively open set \( W \subset S \). If \( \psi = 0 \) on \( W \), then \( \psi = 0 \) on \( B \).

Proof. Assume \( \psi = 0 \) on \( W \). By the reflection principle \( \psi \) has a harmonic extension across \( W \). By Lemma 3.3 it is sufficient to prove \( R \psi \) vanishes everywhere on \( W \).

Let \( \zeta \in W \). It is easily verified that \( R \) commutes with unitary transformations on \( C^n \). Thus we may assume \( \zeta = e := (1,0,\ldots,0) \). Given a point \( z \in C^n \) near \( e \), let \( z' = (\zeta^T z, z_2, \ldots, z_n) \). For \( z' \) near \( 0' \), define \( \phi(z') := \psi(\sqrt{1 - |z'|^2} e + z') \). Put \( D_j = \partial_j \partial z_j \) for \( j = 1,\ldots,2n \) where \( x_{2j-1} = \Re z_j \) and \( x_{2j} = \Im z_j \). Note that \( \phi \) vanishes near \( 0' \), because \( \psi \) vanishes on \( W \). Thus, by a straightforward calculation, we have

\[
0 = D_j \phi(0') = D_j \psi(e)
\]

\[
0 = D^2_j \phi(0') = D^2_j \psi(e) - D_1 \psi(e)
\]

for \( j = 2,\ldots,2n \).

Let \( \eta = (0,1,0,\ldots,0) \); this is the place where we use the hypothesis \( n \geq 2 \). Given \( t \in (0,1) \), consider the function

\[
\psi_{t,\eta}(\lambda) := \psi(te + \lambda \eta)
\]

defined for complex numbers \( \lambda \) such that \( |\lambda| < \sqrt{1 - t^2} \). This function \( \psi_{t,\eta} \) is harmonic near 0, because \( \psi \) is pluriharmonic on \( B \). Thus, computing the Laplacian of \( \psi_{t,\zeta} \) at \( \lambda = 0 \), we obtain

\[
D^2_\zeta \psi(te) + D^2_\zeta \psi(te) = 0.
\]

Since this holds for arbitrary \( t \in (0,1) \), we have by continuity

\[
2D_1 \psi(e) = D^2_\zeta \psi(e) + D^2_\zeta \psi(e) = 0.
\]

Since \( R \psi(e) = D_1 \psi(e) \), this completes the proof. \( \square \)

As a consequence, we obtain the following.

Corollary 3.5 (\( n \geq 2 \)). Let \( u_1,\ldots,u_N \) be pluriharmonic functions on \( B \) and assume \( u_1,\ldots,u_N \in C(B \cup W) \) for some nonempty relatively open set \( W \subset S \). If \( u_1 \cdots u_N = 0 \) on \( W \), then \( u_j = 0 \) on \( B \) for some \( j \).

Proof. Assume \( u_1 \cdots u_N = 0 \) on \( W \). Then one of them, say \( u_j \), vanishes on some nonempty relatively open set \( W \subset S \). Thus \( u_j = 0 \) by Theorem 3.4. \( \square \)

It seems worth mentioning that in the hypotheses of Theorem 3.4 the vanishing property on some nonempty relatively open set cannot be relaxed to the vanishing property on a set of positive \( \sigma \)-measure. In fact it is known (see [25, Theorem 9.3.8]) that one can preassign arbitrary continuous boundary data, except for a set of arbitrarily small \( \sigma \)-measure, to a function pluriharmonic on \( B \) and continuous on \( \overline{B} \). More explicitly, the theorem (on the ball) states that, given \( \phi \in C(S) \) and \( \epsilon > 0 \), there exists some nonconstant function \( f \) holomorphic on \( B \) and continuous on \( \overline{B} \) such that \( \sigma \{ |R f|_S \neq \phi \} < \epsilon \). See also [29, Theorem 15.2] for a more delicate version. So, when \( \phi = 0 \), we see that \( R f|_S = 0 \) outside a set of \( \sigma \)-measure less than \( \epsilon \), but is not identically 0. For such a function \( f \) and \( n \geq 2 \), note that the set \( \{ |R f|_S = 0 \} \) must have empty interior by Theorem 3.4.
Also, for symbols pluriharmonic continuous up to some nonempty relatively open set of $S$, we have the following characterization of compact products on higher dimensional balls. In case of zero products of two Toeplitz operators, rather than compact products, local continuity extension hypothesis will be removed in Theorem 4.6.

**Theorem 3.6** ($n \geq 2$). Let $u_1, \ldots, u_N \in \text{ph}^\infty(S) \cap C(W)$ for some nonempty relatively open set $W \subset S$. If $T_{u_1} \cdots T_{u_N}$ is compact, then $u_j = 0$ a.e. on $S$ for some $j$.

**Proof.** Put $L = T_{u_1} \cdots T_{u_N}$. We see from Theorem 3.2(a) that $P[L]$ continuously extends to $B \cup W$ and $P[L] = u_1 \cdots u_N$ on $W$. On the other hand, since $k_z$ weakly converges to 0 as $|z| \to 1$, we have

$$P[L](z) = \langle Lk_z, k_z \rangle \to 0 \quad \text{as} \quad |z| \to 1$$

and thus $P[L] \in C_0(B)$. It follows that $u_1 \cdots u_N = 0$ on $W$. Meanwhile, since functions $u_1, \ldots, u_N$ are all continuous on $W$, their Poisson-Szegő integrals $P_{u_1}, \ldots, P_{u_N}$ all have continuous extensions to $B \cup W$ and $Pu_j = u_j$ for each $j$. We thus have $Pu_j = 0$ on $B$ for some $j_0$ by Corollary 3.5. Thus $u_j = (Pu_{j_0})^* = 0$ a.e. on $S$. The proof is complete. □

Theorem 3.6 does not extend to the one-dimensional case in general. To see an example, we first recall that semicommutators with continuous symbols are always compact on $T$ which are continuous, nonzero and complete. □

We now close this section with the following remarks.

(1) The higher dimensional phenomenon appearing in Theorem 3.6 has been known in the setting of the polydisk $D^n$ and due to Ding [14, Theorem 3.3]. That is, a product of Toeplitz operators with pluriharmonic symbols, acting on the Hardy space $H^2(D^n)$ over $D^n$ with $n \geq 2$, is compact if and only if at least one of the symbols is zero. In fact Ding proved this theorem for six factors by using the zero product theorem for six factors on $H^2(D)$ due to Gu [19] and Ding's argument now works for arbitrary number of factors thanks to the solution of the finite rank product conjecture due to Aleman and Vukotic [4]. Such a polydisk result raises a natural question whether or not the local continuity extension hypothesis in our Theorem 3.6 is essential.

(2) As a special case of Theorem 3.1, we see that a Toeplitz operator with bounded symbol is compact only when it is the zero operator. Thus, the assertion in Theorem 3.1 is equivalent to that $\sum_{i=1}^{N} \prod_{j=1}^{M} T_{u_{ij}}$ is a compact perturbation of a Toeplitz operator only when it is a compact perturbation of $T_{\sum_{i=1}^{N} \prod_{j=1}^{M} u_{ij}}$. The same result has been known on the disk by Douglas [15] and on the polydisk by Ding [14, Theorem 3.1]. Also, Guo and Zheng [22] showed on the disk that an operator $L$ of the type under consideration is a compact perturbation of a Toeplitz operator if and only if

$$\|L - T_{\varphi_2}^*LT_{\varphi_1}\| \to 0$$

as $|a| \to 1$. More recently, Xia [31] proved that a general bounded linear operator $L$ on $H^2(D)$ is a compact perturbation of a Toeplitz operator if and only if $L - T_uLT_u$ is compact for every inner function $u$, settling a long-standing problem of Douglas [16, Problem 7.38].
(3) As another special case we see that \( \sum_{i=1}^{N} T_{u_j} T_{v_j} \) is a compact perturbation of a Toeplitz operator if and only if the associated sum \( \sum_{j=1}^{N} (T_{u_j} v_j - T_{u_j} T_{v_j}) \) of semi-commutators is compact.

4. Finite Rank Operators

In the setting of the Hardy space over the polydisk, the last-named author [23] obtained a characterization for commuting Toeplitz operators with one of the symbols pluriharmonic and the other arbitrary. Motivated by such a result, we here consider operators of more general type. Namely, we consider operators \( L \) of the form

\[
(4.1) \quad L = \sum_{j=1}^{N} T_{u_j} T_{v_j}
\]

where either \( u_j \in \text{ph}^\infty(B) \) or \( v_j \in \text{ph}^\infty(B) \) for each \( j \).

We recall some basic facts about projections of certain type of functions. Note that the projection \( Q \) naturally extends via the integral representation (1.1) to an integral operator from \( L^1(S) \) into the space of all holomorphic functions on \( B \). It is easily seen that

\[
(4.2) \quad Qf = f \quad \text{and} \quad Q(\overline{f} K_a) = \overline{f(a)} K_a
\]

for \( f \in H^1(B) \) and \( a \in B \). We also need the simple fact that

\[
(4.3) \quad Q[\overline{f} (Qu)] = Q(f u)
\]

for \( f \in H^2(B) \) and \( u \in L^2(S) \). Finally, we recall the well-known growth rate of Hardy functions. Namely, given \( 0 < p < \infty \), we have

\[
(4.4) \quad \lim_{|z| \to 1} (1 - |z|^2)^{n/p} |f(z)| = 0
\]

for \( f \in H^p(B) \); see [28, Theorem 7.2.5].

We begin with a general observation for operators as in (1.2) to be finite rank operators. In what follows the notation \( x \otimes y \) with \( x, y \in H^2(B) \) stands for the operator defined by

\[
(x \otimes y)f = \langle f, y \rangle x
\]

for \( f \in H^2(B) \). Also, the letter \( z \) will often denote the identity map of \( B \).

Lemma 4.1. Let \( L \) be as in (1.2) and \( x_j, y_j \in H^2(B) \) for \( j = 1, \ldots, J \). Then

\[
(4.5) \quad L = \sum_{j=1}^{J} x_j \otimes y_j
\]

holds if and only if the following two conditions hold:

(a) \( \sum_{i=1}^{N} \prod_{j=1}^{M} u_{ij} = 0 \) a.e. on \( S \);

(b) \( P[L] - (1 - |z|^2)^n \sum_{j=1}^{J} x_j \overline{y_j} \) is \( \mathcal{M} \)-harmonic on \( B \).
Proof. First suppose that (4.5) holds. We have (a) by Theorem 3.1. Note that
\[ K_z(z) = (1 - |z|^2)^{-n} \quad \text{and thus} \quad P[x \otimes y] = (1 - |z|^2)^n x^\gamma \text{ for } x, y \in H^2(B). \]
So, taking Poisson-Szegö transforms of both sides of (4.5), we have
\[ P[L] = (1 - |z|^2)^n \sum_{j=1}^{J} x_j \overline{y_j} \quad \text{on } B, \]
which implies (b).

Conversely, assume (a) and (b). Put
\[ \psi_j := \frac{1}{P[L]} - (1 - |z|^2)^n \sum_{j=1}^{J} x_j \overline{y_j}. \]
Note that \( \psi \) is bounded by (4.4). Being a bounded \( \mathcal{M} \)-harmonic function, \( \psi \) is recovered by the Poisson-Szegö integral of its boundary function \( \psi^* \); see [28, Theorem 4.3.3]. Meanwhile, we have by (4.4) and Theorem 3.2(b)
\[ \psi^* = P[L]^* = \sum_{i=1}^{N} \prod_{j=1}^{M} u_i v_j = 0 \quad \text{a.e. on } S; \]
we used assumption (a) for the last equality. Accordingly, we have \( \psi = 0 \), which means
\[ P[L] = P \left[ \sum_{j=1}^{J} x_j \otimes y_j \right] \quad \text{on } B. \]
Now, since the Poisson-Szegö transform is one-to-one, we conclude (4.5). The proof is complete. \( \square \)

The next theorem generalizes the main result (over the polydisk) of [23]. Moreover, the present argument based on Lemma 4.1 is completely different from and much simpler than that in [23].

**Theorem 4.2.** Assume either \( u_j \in \text{ph}^\infty(B) \) or \( v_j \in \text{ph}^\infty(B) \) for \( j = 1, \ldots, N \) and let \( x_j, y_j \in H^2(B) \) for \( j = 1, \ldots, M \). Put
\[ \psi_j := \begin{cases} Qu_j P v_j & \text{if } u_j \in \text{ph}^\infty(B) \\ Pu_j Q v_j & \text{if } v_j \in \text{ph}^\infty(B) \end{cases} \]
for \( j = 1, \ldots, N \). Then
\[ \sum_{j=1}^{N} T_u T_{v_j} = \sum_{j=1}^{M} x_j \otimes y_j \quad \text{(4.6)} \]
holds if and only if the following two conditions hold:

(a) \[ \sum_{j=1}^{N} u_j v_j = 0 \quad \text{a.e. on } S; \]

(b) \[ \sum_{j=1}^{N} \psi_j - (1 - |z|^2)^n \sum_{j=1}^{M} x_j \overline{y_j} \text{ is } \mathcal{M}-\text{harmonic on } B. \]
Proof. By Lemma 4.1, we only need to prove that (b) holds if and only if
\[
P \left[ \sum_{j=1}^{N} T_{u_j} T_{v_j} \right] - (1 - |z|^2)^n \sum_{j=1}^{M} x_j \bar{y}_j \text{ is } \mathcal{M}\text{-harmonic.}
\]

Thus it suffices to prove that
\[
P \left[ \sum_{j=1}^{N} T_{u_j} T_{v_j} \right] - \sum_{j=1}^{N} \psi_j \text{ is } \mathcal{M}\text{-harmonic.}
\]

In order to prove this we prove that \(P \left[ T_{u_j} T_{v_j} \right] - \psi_j \text{ is } \mathcal{M}\text{-harmonic for each } j\).

Fix \(j\). First, consider the case where \(u_j \in p_{h^\infty}(B)\). Let \(f_j\) and \(\overline{f_j}\) be the holomorphic part and co-holomorphic part of \(u_j\), respectively, so that \(u_j = f_j + \overline{f_j}\). Let \(z \in B\) be an arbitrary point. We then have by (4.2) and (4.3)
\[
T_{u_j} T_{v_j} k_z = f_j Q(v_j k_z) + Q(\overline{f_j} v_j k_z)
\]
so that
\[
P[T_{u_j} T_{v_j}](z) = \langle f_j Q(v_j k_z), k_z \rangle + \langle Q(\overline{f_j} v_j k_z), k_z \rangle
\]
\[
= f_j(z) Q(v_j k_z)(z) \| K_z \|_2^{1/2} + P[\overline{f_j} v_j](z).
\]
This shows that \(P[T_{u_j} T_{v_j}] - \psi_j\) is a Poisson-Szegő integral and hence \(\mathcal{M}\)-harmonic, as asserted.

Next, consider the case where \(v_j \in p_{h^\infty}(B)\). Let \(h_j\) and \(\overline{h_j}\) be the holomorphic part and co-holomorphic part of \(v_j\), respectively, so that \(v_j = h_j + \overline{h_j}\). We then have by (4.2)
\[
T_{u_j} T_{v_j} k_z = Q(u_j h_j k_z) + \overline{h_j(z)} Q(u_j k_z)
\]
and thus
\[
P[T_{u_j} T_{v_j}](z) = \langle Q(u_j h_j k_z) + \overline{h_j(z)} Q(u_j k_z), k_z \rangle
\]
\[
= P[u_j h_j](z) + \overline{h_j(z)} P u_j(z).
\]
This again shows that \(P[T_{u_j} T_{v_j}] - \psi_j\) is a Poisson-Szegő integral and hence \(\mathcal{M}\)-harmonic, as asserted. The proof is complete. \(\square\)

In case all the symbols in Theorem 4.2 are pluriharmonic, the characterization reduces as in the next corollary.

Corollary 4.3. Let \(u_j, v_j \in p_{h^\infty}(B)\) for \(j = 1, \ldots, N\) and \(x_j, y_j \in H^2(B)\) for \(j = 1, \ldots, M\). Let \(f_j, k_j\) be the holomorphic parts of \(u_j, \overline{v_j}\), respectively, for \(j = 1, \ldots, N\). Let \(\lambda \in L^\infty(S)\). Then
\[
\sum_{j=1}^{N} T_{u_j} T_{v_j} = T_{\lambda} + \sum_{j=1}^{M} x_j \otimes y_j
\]
holds if and only if the following two conditions hold:

(a) \(\sum_{j=1}^{N} u_j v_j = \lambda \text{ a.e. on } S\);

(b) \(\sum_{j=1}^{N} f_j k_j - (1 - |z|^2)^n \sum_{j=1}^{M} x_j \bar{y}_j \text{ is } \mathcal{M}\text{-harmonic on } B\).
Proof. Note $T_\lambda = T_1 T_\lambda$ where $T_1$ denotes the Toeplitz operator whose symbol is the constant function $1$. Let $g_j$ and $h_j$ be the holomorphic parts of $v_j$ and $v_j$, respectively, so that $u_j = f_j + \overline{g_j}$ and $v_j = h_j + \overline{k_j}$ for each $j = 1, \ldots, N$. Since $Q u_j = f_j + \overline{g_j(0)}$ and $P v_j = h_j + \overline{k_j}$ for each $j$, we see that the difference

$$\sum_{j=1}^N Qu_j Pv_j - \sum_{j=1}^N f_j h_j + g_j(0)h_j + g_j(0)k_j$$

is pluriharmonic and thus $\mathcal{M}$-harmonic. So, the theorem holds by Theorem 4.2. \qed

As an immediate consequence of Theorem 4.2, we have a characterization for finite rank sum of semi-commutators with pluriharmonic symbols as in the next corollary. Given Toeplitz operators $T_u$ and $T_v$, we let

$$[T_u, T_v] = T_{uv} - T_u T_v$$

denote the semi-commutator.

**Corollary 4.4.** Under the hypotheses and notation of Corollary 4.3

$$\sum_{j=1}^N [T_{u_j}, T_{v_j}] = \sum_{j=1}^M x_j \otimes y_j$$

holds if and only if

$$\sum_{j=1}^N f_j k_j + (1 - |z|^2)^n \sum_{j=1}^M x_j \overline{y_j} \text{ is } \mathcal{M}\text{-harmonic on } \mathcal{B}.$$

In the Bergman space case, it is known that if a sum of finitely many semi-commutators of Toeplitz operators with pluriharmonic symbols has finite rank, then it is already the zero operator; see [8], [21] (over the disk) and [10] (over the polydisk). So, it seems worth mentioning explicit examples showing that the situation on the Hardy space is different.

**Examples.** (1) Pick any polynomial $p(z, \overline{z}) := \sum_{\alpha, \beta} a_{\alpha \beta} z^\alpha \overline{z}^\beta$ on $\mathbb{C}^n$ and let $b_{\alpha \beta} = b_{\alpha \beta}(p)$ be the coefficients determined by the identity

$$(1 - |z|^2)^n p(z, \overline{z}) = -\sum_{\alpha, \beta} b_{\alpha \beta} z^\alpha \overline{z}^\beta.$$

Here, we are using the conventional multi-index notation. Note that $\sum_{|\alpha| + |\beta| = 0} b_{\alpha \beta} z^\alpha \overline{z}^\beta$ is pluriharmonic. Thus we have by Corollary 4.4

$$\sum_{|\alpha| + |\beta| > 0} b_{\alpha \beta}(T_{z^\alpha}, T_{\overline{z}^\beta}) = \sum_{\alpha, \beta} a_{\alpha \beta}(z^\alpha \otimes z^\beta).$$

In general we have

$$\sum_{\alpha, \beta} b_{\alpha \beta}(T_{z^\alpha x}, T_{z^\beta \overline{y}}) = \sum_{\alpha, \beta} a_{\alpha \beta}(z^\alpha x \otimes z^\beta y)$$

for any $x, y \in H^\infty(\mathcal{B})$. So, an arbitrary positive integer can be the rank of a sum of finitely many semi-commutators with pluriharmonic symbols. In particular, the constant polynomial $p = 1$ yields the rank-one operator

$$\sum_{|\alpha| \leq n} c_{\alpha}(T_{z^\alpha x}, T_{z^\beta \overline{y}}) = x \otimes y$$
where $c_\alpha = b_{\alpha \alpha}(1)$. Using this, we see that any finite rank operator generated by functions in $H^\infty(B)$ can be represented by a sum of finitely many semi-commutators with pluriharmonic symbols.

(2) Let's consider the special case $n = 1$. Let $N$ be an arbitrary positive integer. Since

$$ (1 - |z|^2) \sum_{j=1}^{N-1} |z|^2 = 1 - |z|^{2N}, $$

we have by (4.7)

$$ (T_{z^N x}, T_{z^N y}) - (T_x, T_y) = \sum_{j=0}^{N-1} z^j x \otimes z^j y $$

for any $x, y \in H^\infty(D)$. In particular, we have

$$ (T_{z^N x}, T_{z^N}) = \sum_{j=0}^{N-1} z^j x \otimes z^j $$

for any $x \in H^\infty(D)$. This shows that an arbitrary positive integer can be the rank of just one single semi-commutator. To see more examples with such property, one may verify via Corollary 4.4

$$ (T_{(z^N - a)x}, T_{z^N}) = \sum_{j=0}^{N-1} z^j x \otimes \frac{z^j}{1 - az^N}, \quad a \in \mathbb{D}, $$

which reduces to (4.8) if $a = 0$.

The following remarks are in order in conjunction with our results above.

1. Since $ph^\infty(T) = L^\infty(T)$, the one-variable case of Theorem 4.2 and its corollaries apply, when $M$-harmonic is replaced by harmonic, to arbitrary bounded symbols.

2. Ding and Zheng [17, Theorem 3.1] have recently obtained a characterization for finite rank sum of two semi-commutators on $H^2(D)$. Their study was motivated by the Axler-Chang-Sarason theorem ([3]) asserting that the semi-commutator $[T_u, T_v]$ has finite rank if and only if either one of associated Hankel operators $H_u$ and $H_v$ has finite rank; we refer to [17] for definition Hankel operators on $H^2(D)$. Such a characterization is associated with the characterization of finite rank Hankel operators due to Kronecker (see [26]) asserting that $H_u$ has finite rank if and only if $pu \in H^\infty(D)$ for some nonzero holomorphic polynomial $p$. The characterization in [17] is of the form that generalizes the Kronecker theorem. We remark that our characterization (the case $N \leq 2$ of Corollary 4.4) is of the form completely different from the Axler-Chang-Sarason theorem or the one by Ding and Zheng.

3. For a finite sum of Toeplitz products with pluriharmonic symbols acting on the Bergman space over the disk or polydisk, the authors obtained characterizations for compactness; see [8] and [10]. This naturally gives rise to the problem to characterize compactness for operators considered either in Theorem 4.2 or its corollaries over higher dimensional balls. By Theorem 3.1 this problem is the same as the one to characterize compact sums of finitely many semi-commutators of Toeplitz operators for which either one (or both) of symbols is pluriharmonic. On the disk, several characterizations have been known and due to Gu and Zheng [20, Theorem 10]. Note also that the disk case is contained in the characterization (3.3) for more general operators as in (1.2). Their proofs, however,
depend on several properties restricted to the one-variable case and the ball case appears to be open.

We now apply our results and thereby obtain more concrete characterizations for two special types of operators with pluriharmonic symbols to be the zero operator. The first one is the zero product of two Toeplitz operators and the second one the zero sum of two semi-commutators.

In order to characterize the zero product of two Toeplitz operators with pluriharmonic symbols, we need the following lemma.

**Lemma 4.5.** Let \( f \in H^{2n}(B) \) and assume that \( g \) is a nonconstant holomorphic function on \( B \). If \( fg \) is \( \mathcal{M} \)-harmonic on \( B \), then \( f \) is constant.

In fact we will apply the above lemma when both \( f \) and \( g \) belong to \( H^{2n}(B) \). In such a case, the above lemma is known and due to Zheng [32, Theorem 6]. Here, mainly for completeness, we state the lemma as above under a slightly weakened hypothesis and provide below a proof based on the characterization of \( \mathcal{M} \)-harmonic products due to Ahern and Rudin [2, Theorem 1].

**Proof.** Before proceeding, we introduce some temporary notation to be used in this proof. We denote by \( A \) the area measure on \( D \) normalized to have total mass 1. We also attach subscripts to \( S \) and \( \sigma \) to distinguish dimensions involved.

Assume that \( fg \) is \( \mathcal{M} \)-harmonic on \( B \). To reach a contradiction, assume that \( f \) is non-constant. Then, by the characterization of Ahern and Rudin mentioned above, we have \( n \geq 3 \) and there exist an integer \( m \) with \( 2 \leq m \leq n - 1 \), a unitary transformation \( U \), and nonconstant entire functions \( F \) on \( \mathbb{C}^{m-1} \) and \( G \) on \( \mathbb{C}^{n-m} \) such that

\[
f(Uz) = F\left(\frac{z_2}{1-z_1}, \ldots, \frac{z_m}{1-z_1}\right) \quad \text{and} \quad g(Uz) = G\left(\frac{z_{m+1}}{1-z_1}, \ldots, \frac{z_n}{1-z_1}\right)
\]

for \( z \in B \). We may assume \( f(0) = F(0) = 0 \). We will deduce that \( F \) is identically 0, which is a desired contradiction.

Let \( F = \sum_{s=1}^{\infty} F_s \) be the homogeneous expansion of \( F \) where each \( F_s \) is a homogeneous polynomial on \( \mathbb{C}^{m-1} \) of degree \( s \). The homogeneous expansion of \( F^n \) is then given by \( \sum_{s=n}^{\infty} \psi_s \) where \( \psi_s = \sum_{s_1 + \cdots + s_n = s} F_{s_1} \cdots F_{s_n} \). It follows from homogeneity that

\[
f^n(Uz) = \sum_{s=n}^{\infty} \psi_s(z_2, \ldots, z_m) \quad \text{and} \quad g(Uz) = \sum_{s=n}^{\infty} \psi_s(z_{m+1}, \ldots, z_n).
\]

Note that the functions \( z_k^l \psi_s(z_2, \ldots, z_m) \), when considered as functions in \( H^2(B) \), are mutually orthogonal. We thus have

\[
\|f^n\|^2_{H^2} = \sum_{s=n}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma^2(s+k)}{(k!)^2 \Gamma^2(s)} \int_{S^n} |z_k^l \psi_s(\zeta_2, \ldots, \zeta_m)|^2 \, d\sigma_n(\zeta).
\]
Moreover, by the slice integration formula ([34, Lemma 1.10]), the integral above is equal to
\[
\int_{S_n} |ζ_s^{(j)}(ζ_2, \ldots, ζ_n)|^2 dσ_n(ζ)
\]
\[= (n - 1) \int_{S_{n-1}} |φ_s(\sqrt{1 - |λ|^2} \eta)|^2 \left\{ \int_D (1 - |λ|^2)^{n-2} |λ|^{2k} dA(λ) \right\} dσ_{n-1}(η)
\]
\[= (n - 1) \int_{S_{n-1}} |φ_s(η)|^2 dσ_{n-1}(η) \int_D (1 - |λ|^2)^{n-2+s} |λ|^{2k} dA(λ).
\]

Also, note that we have
\[
\sum_{k=0}^{\infty} \frac{Γ^2(s + k)}{(k!)^2 2Γ^2(s)} \int_D (1 - |λ|^2)^{n-2+s} |λ|^{2k} dA(λ) = \int_D (1 - |λ|^2)^{n-2+s} dA(λ)
\]
for each \( s \geq n \) by orthogonality of one-variable monomials with respect to the radial measure \((1 - |λ|^2)^{n-2+s} dA(λ)\). Combining these observations, we obtain
\[
\|f^n\|^2_{H^2} = (n - 1) \sum_{s=n}^{\infty} \int_{S_{n-1}} |φ_s(η)|^2 dσ_{n-1}(η) \int_D (1 - |λ|^2)^{n-2+s} |λ|^{2k} dA(λ).
\]

Note that the second integral factor of the above diverges by [34, Theorem 1.12], because \(2s - (n + s) = s - n \geq 0\). Now, since \(f^n \in H^2(B)\), the first integral factor of the above must vanish. So, we have \(ψ_s = 0\) for each \( s \geq n \) and thus conclude \(F = 0\), as asserted.

The proof is complete. \(□\)

The following characterization shows that the product of two Toeplitz operators with pluriharmonic symbols can be the zero operator only in the trivial case. The one-dimensional case is contained in the zero product theorem of Aleman and Vucotti [4].

**Theorem 4.6.** Let \(u, v \in ph^∞(B)\). Then \(T_uT_v = 0\) if and only if either \(u = 0\) or \(v = 0\).

**Proof.** The sufficiency being trivial, we only need to prove the necessity. Assume \(T_uT_v = 0\). By Corollary 4.3 we have (i) \(uv = 0\) a.e. on \(S\); and (ii) \(f\overline{g}\) is \(M\)-harmonic on \(B\) where \(f\) is the holomorphic part of \(u\) and \(\overline{g}\) is the co-holomorphic part of \(v\). Recall that \(f, g \in H^p(B)\) for all \(p < ∞\). Thus we see from (ii) and Lemma 4.5 that either \(f\) or \(g\) is constant, i.e., either \(\overline{f}\) or \(v\) is holomorphic. If \(\overline{f}\) is holomorphic and \(u \neq 0\) on \(B\), then \(v = 0\) a.e. on \(B\) by (i) and hence \(uv = 0\) on \(B\). Also, if \(v\) is holomorphic and \(u \neq 0\), then \(uv = 0\) a.e. on \(B\) by (i) and hence \(u = 0\) on \(B\). Thus we conclude that either \(u = 0\) or \(v = 0\). The proof is complete. \(□\)

An application of Corollary 4.4 and Lemma 4.5 provides another proof of the well-known fact (see [32, Theorem C]) that if \(\{T_u, T_v\} = 0\) where \(u, v \in ph^∞(B)\), then either \(\overline{u}\) or \(v\) is holomorphic (the converse is also true and trivial). Here, based on Corollary 4.4, we proceed to characterize in a similar way the zero sum of two semi-commutators with pluriharmonic symbols. The following fact is useful for our purpose.

**Lemma 4.7.** Let \(f_j, g_j\) be holomorphic functions on \(B\) for \(j = 1, \ldots, N\). Then \(\sum_{j=1}^{N} f_j \overline{g_j}\) is pluriharmonic on \(B\) if and only if the equality
\[
\sum_{j=1}^{N} [f_j(z) - f_j(0)][\overline{g_j(w)} - g_j(0)] = 0
\]
holds for all \(z, w \in B\).
Proof. In case \( n = 1 \) the lemma is the content of [8, Theorem 3.3]. One may conclude the lemma for general dimension by applying such a one-variable result to slice functions \( f_\zeta(\lambda) := f(\lambda \zeta) \) and \( g_\zeta(\lambda) := g(\lambda \zeta) \) where \( \zeta \in \mathbb{S} \) and \( \lambda \in \mathbb{D} \).

We now prove the following characterization for the zero sum of two semi-commutators with pluriharmonic symbols.

**Theorem 4.8.** Let \( u_1, u_2, v_1, v_2 \in \mathbb{p}h^\infty(\mathcal{B}) \) and assume that \( (T_{u_1}, T_{v_1}) \) and \( (T_{u_2}, T_{v_2}) \) are both nonzero. Let \( f_j \) and \( \overline{f_j} \) be the holomorphic part of \( u_j \) and the co-holomorphic part of \( v_j \), respectively, for \( j = 1, 2 \). Then the following statements are equivalent:

(a) \( (T_{u_1}, T_{v_1}) + (T_{u_2}, T_{v_2}) = 0; \)

(b) \( f_1 \overline{f_1} + f_2 \overline{f_2} \) is \( \mathcal{M} \)-harmonic on \( \mathcal{B}; \)

(c) \( f_1 \overline{f_1} + f_2 \overline{f_2} \) is pluriharmonic on \( \mathcal{B}; \)

(d) There are nonzero constants \( a_1, a_2, b_1, b_2 \) such that \( a_1b_1 + a_2b_2 = 0 \) and

\[
(4.9)
\]

\[
a_1 u_1 + a_2 u_2 \in H^\infty(\mathcal{B}), \quad b_1 v_1 + b_2 v_2 \in H^\infty(\mathcal{B}).
\]

Zheng [33, Corollary 5.18] proved the equivalence (b) \( \iff \) (c) of the above theorem in the course of a complete description of the zero commutator with pluriharmonic symbols. While one may probably extend the idea of [33] to prove other equivalences, we provide below a proof relying on Corollary 4.4 and Lemma 4.7. We also remark that the disk version of the equivalence (a) \( \iff \) (d) is contained (in a different form) in [17, Theorem 3.1].

Proof. The equivalence (a) \( \iff \) (b) holds by Corollary 4.4. The equivalence (b) \( \iff \) (c) is known, as mentioned above. Now we prove that (a), together with (c), implies (d). So, assume (a) and (c). Since \( (T_{u_j}, T_{v_j}) \neq 0 \) for each \( j \), none of \( \overline{f_j} \) and \( v_j \) is holomorphic. Since \( f_1 \overline{f_1} + f_2 \overline{f_2} \) is pluriharmonic on \( \mathcal{B} \), we have by Lemma 4.7

\[
\sum_{j=1}^{2} (f_j(z) - f_j(0))(\overline{g_j(w)} - g_j(0)) = 0, \quad z, w \in \mathcal{B}.
\]

Note that this can be written as

\[
(4.10)
\]

\[
\sum_{j=1}^{2} (\overline{\overline{f_j}} - \overline{Q f_j})(z)(v_j - Q v_j)(w) = 0
\]

where we used the same letter \( u_j \) and \( v_j \), for functions on \( \mathbb{S} \) and their pluriharmonic extensions on \( \mathcal{B} \). Since \( \overline{Q f_j} \neq \overline{Q f_j} \) for each \( j \), there is some \( z_0 \in \mathcal{B} \) such that \( u_j(z_0) \neq Q v_j(z_0) \) for each \( j \). Thus, setting \( b_j := u_j(z_0) - Q v_j(z_0) \neq 0 \), we have by (4.10)

\[
b_1 v_1 + b_2 v_2 = \sum_{j=1}^{2} b_j Q v_j \in H^\infty(\mathcal{B}).
\]

This proves (4.9) for \( v_j \)'s. Similarly, we have nonzero constants \( a_1 \) and \( a_2 \) with which (4.9) holds for \( u_j \)'s. Now, by (4.9), we have \( a_1 u_1 = \overline{f} - a_2 v_2 \) and \( b_1 v_1 = g - b_2 v_2 \) for some \( f, g \in H^\infty(\mathbb{S}) \). We thus have

\[
a_1 b_1 (T_{u_1}, T_{v_1}) = (T_{u_1}, T_{b_1 v_1}) = (T_{\overline{f}} - a_2 T_{u_2}, T_g - b_2 T_{v_2}) = a_2 b_2 (T_{u_2}, T_{v_2})
\]

so that (a) yields

\[
(4.10) (a_1 b_1 + a_2 b_2)(T_{u_2}, T_{v_2}) = 0.
\]
This yields \(a_1b_1 + a_2b_2 = 0\), as required, and thus (d) holds. Finally, one may go backwards along the proof just completed and show the implication \( (d) \implies (a) \). The proof is complete.

5. Other settings: Polydisk and Bergman space

In this section we consider the possibility of extending some of our results to other settings such as the Hardy space over the polydisk and the Bergman space over the ball.

5.1. Polydisk versions. Let \(D^n\) be the unit polydisk in \(C^n\). All the basic notions discussed on the Hardy space over the ball have counterparts on the Hardy space over the polydisk. For simplicity we carry the notation relevant to the Hardy space over the ball over to the polydisk setting, unless otherwise specified.

The Hardy space \(H^2(D^n)\) over \(D^n\) is the Hilbert space of all holomorphic functions \(f\) on \(D^n\) such that

\[
\|f\|_{H^2} := \sup_{0 < r < 1} \left\{ \int_{T^n} |f(r\zeta)|^2 \, d\sigma(\zeta) \right\}^{1/2} < \infty;
\]

this time \(\sigma\) denotes the normalized Haar measure on \(T^n\). As in the ball case, the space \(H^2(D^n)\) is isometrically identified with a closed subspace of \(L^2(T^n) = L^2(T^n, \sigma)\) via (nontangential) boundary functions.

We continue using the same notation for the reproducing kernel \(K_z\) for \(H^2(D^n)\), the normalized kernel \(k_z\), the Hilbert space orthogonal projection \(Q : L^2(T^n) \to H^2(D^n)\), the inner product \(\langle \cdot, \cdot \rangle\) on \(L^2(T^n)\), and the Toeplitz operator \(T_u\) with symbol \(u \in L^\infty(T^n)\) on \(H^2(D^n)\). So, given \(z \in D^n\), we have

\[
K_z(\zeta) = \prod_{j=1}^n \left( 1 - \zeta_j \overline{\zeta_j} \right)^{-1}
\]

for \(\zeta \in T^n\), \(k_z = K_z/\|K_z\|_{H^2}\),

\[
Q\psi(z) = \langle \psi, K_z \rangle \quad \text{and} \quad T_u f = Q(u f)
\]

for \(\psi \in L^2(T^n)\) and \(f \in H^2(D^n)\). We also use the same letter \(P\) for the Poisson integral of integrable functions or the Poisson transform of bounded linear operators. So, we have

\[
P u(z) = \langle u k_z, k_z \rangle \quad \text{and} \quad P[L](z) = (L k_z, k_z), \quad z \in D^n
\]

for \(u \in L^1(T^n)\) and bounded linear operators \(L\) on \(H^2(D^n)\). Again the Poisson transform is one-to-one and we have \(P[T_u] = P u\) for \(u \in L^\infty(T^n)\). Conjugate operators \(L_z\) are defined exactly the same way as in the case of the ball, i.e.,

\[
L_z = U_z L U_z
\]

where \(U_z\) is the unitary weighted composition operator adjusted to the polydisk setting.

We say that a function on \(D^n\) is \(n\text{-harmonic}\) if it is harmonic in each variable separately. Note that Poisson integrals of functions are always \(n\text{-harmonic}\). The notion of \(n\text{-harmonicity on } D^n\) is the one that corresponds to \(\mathcal{M}\text{-harmonicity on } B\).

We first prove the polydisk analogue of Lemma 4.1.

**Lemma 5.1.** Let \(L\) be as in (1.2) and \(x_j, y_j \in H^2(D^n)\) for \(j = 1, \ldots, J\). Then

\[
L = \sum_{j=1}^J x_j \otimes y_j
\]

holds if and only if the following two conditions hold:

\[
(a) \sum_{i=1}^N \prod_{j=1}^M u_{ij} = 0 \ a.e \ on \ T^n;
\]
(b) $P[L] - \prod_{i=1}^{n}(1 - |z_i|^2) \sum_{j=1}^{J} x_j y_j$ is n-harmonic on $D^n$.

Proof. First suppose that (5.1) holds. We have (a) by Theorem 3.1 of [14]. Note $P[x \otimes y] = \prod_{i=1}^{n}(1 - |z_i|^2)^n x \otimes y$ for $x, y \in H^2(D^n)$. So, taking Poisson transforms of both sides of (5.1), we have

$$P[L] = \prod_{i=1}^{n}(1 - |z_i|^2) \sum_{j=1}^{J} x_j y_j$$
on D^n,$$

which implies (b).

Conversely, assume (a) and (b). Put

$$\psi := P[L] - \prod_{i=1}^{n}(1 - |z_i|^2) \sum_{j=1}^{J} x_j y_j.$$

Note that $\psi$ is bounded by (4.4). Being a bounded n-harmonic function, $\psi$ is recovered by the Poisson integral of its boundary function $\psi^*$; see [27, Section 2]. By the same argument as in Theorem 3.2, we see

$$\psi^* = P[L]^* = \sum_{j=1}^{N} \prod_{i=1}^{M} u_j v_j.$$

It follows from (4.4) and (a) that $\psi^* = 0$ a.e. on $T^n$. Accordingly, we have $\psi = 0$, which means

$$P[L] = P \left[ \sum_{j=1}^{J} x_j \otimes y_j \right] \text{ on } D^n.$$

Now, since the Poisson transform is one-to-one, we conclude (5.1). The proof is complete. $\square$

Having Lemma 5.1, we have the following polydisk analogue of Theorem 4.2 by exactly the same argument.

**Theorem 5.2.** Assume either $u_j \in ph^\infty(D^n)$ or $v_j \in ph^\infty(D^n)$ for $j = 1, \ldots, N$ and let $x_j, y_j \in H^2(D^n)$ for $j = 1, \ldots, M$. Put

$$\psi_j := \begin{cases} Q u_j P v_j & \text{if } u_j \in ph^\infty(D^n) \\ P u_j Q v_j & \text{if } v_j \in ph^\infty(D^n) \end{cases}$$

for $j = 1, \ldots, N$. Then

$$\sum_{j=1}^{N} T_{u_j} T_{v_j} = \sum_{j=1}^{M} x_j \otimes y_j$$

(5.2) holds if and only if the following two conditions hold:

(a) $\sum_{j=1}^{N} u_j v_j = 0$ a.e on $T^n$;

(b) $\sum_{j=1}^{N} \psi_j - \prod_{i=1}^{n}(1 - |z_i|^2) \sum_{j=1}^{M} x_j y_j$ is n-harmonic on $D^n$.

Also, we have the corresponding results of Corollaries 4.3 and 4.4.
Corollary 5.3. Let $u_j, v_j \in \mathcal{H}^\infty(\mathbb{D}^n)$ for $j = 1, \ldots, N$ and $x_j, y_j \in H^2(\mathbb{D}^n)$ for $j = 1, \ldots, M$. Let $f_j, k_j$ be the holomorphic parts of $u_j, \overline{v_j}$, respectively, for $j = 1, \ldots, N$. Let $\lambda \in L^\infty(\mathbb{T}^n)$. Then

$$\sum_{j=1}^N T_{u_j} T_{v_j} = T_\lambda + \sum_{j=1}^M x_j \otimes y_j$$

holds if and only if the following two conditions hold:

(a) $\sum_{j=1}^N u_j v_j = \lambda \text{ a.e on } \mathbb{T}^n$;

(b) $\sum_{j=1}^M f_j k_j - \prod_{i=1}^n (1 - |z_i|^2) \sum_{j=1}^M x_j \overline{y_j}$ is $n$-harmonic on $\mathbb{D}^n$.

Corollary 5.4. Under the hypotheses and notation of Corollary 5.3

$$\sum_{j=1}^N [T_{u_j}, T_{v_j}] = \sum_{j=1}^M x_j \otimes y_j$$

holds if and only if

$$\sum_{j=1}^M f_j k_j + \prod_{i=1}^n (1 - |z_i|^2) \sum_{j=1}^M x_j \overline{y_j} \text{ is } n\text{-harmonic on } \mathbb{D}^n.$$ 

5.2. Compact products on the Bergman space. The Bergman space $A^2(\mathcal{B})$ over $\mathcal{B}$ is the Hilbert space of all holomorphic functions in $L^2(\mathcal{B}) = L^2(\mathcal{B}, V)$ where $V$ is the volume measure on $\mathcal{B}$ normalized to have total mass 1.

All the basic notions discussed in the Hardy space case also have Bergman space analogues. For simplicity again we continue using the same notation for the reproducing kernel $K_z$ for $A^2(\mathcal{B})$, the normalized kernel $k_z$, the Hilbert space orthogonal projection $Q : L^2(\mathcal{B}) \rightarrow A^2(\mathcal{B})$, the inner product $(\cdot, \cdot)$ on $L^2(\mathcal{B})$, and the Toeplitz operator $T_u$ with symbol $u \in L^\infty(\mathcal{B})$ on $A^2(\mathcal{B})$. So, given $z \in \mathcal{B}$, we have $K_z(w) = (1 - w \cdot \overline{z})^{-(n+1)}$ for $w \in \mathcal{B}$, $k_z = K_z/\|K_z\|_{A^2}$ and

$$Q\psi(z) = (\psi, K_z) \quad \text{and} \quad T_u f = Q(u f)$$

for $\psi \in L^2(\mathcal{B})$ and $f \in A^2(\mathcal{B})$.

Given a bounded linear operator $L$ on $A^2(\mathcal{B})$, the Berezin transform $B[L]$, the notion that corresponds to the Poisson-Szegö transform, and the conjugate operator $L_z$ are defined exactly the same way as in the case of the Hardy space. More explicitly, we have

$$B[L](z) = (L k_z, k_z) \quad \text{and} \quad L_z = U_z L U_z$$

where $U_z$ is the unitary weighted composition operator adjusted to the Bergman space setting.

The Bergman space analogue of Theorem 3.2(a) is proved in [7, Proposition 2.1]. For the Bergman space analogue of Theorem 3.2(b), we have a bit stronger result as in the first part of the next theorem.

Theorem 5.5. For $u_1, \ldots, u_N \in L^\infty(\mathcal{B})$, let $L = T_{u_1} \cdots T_{u_N}$. Then the following statements hold.

(a) If $u_j$ has an admissible limits at $\zeta \in S$ for each $j$, then $L_z \rightarrow (u_1 \cdots u_N)^*(\zeta)$ in the strong operator topology as $z \rightarrow \zeta$ admissibly.
(b) If \( u_j \in ph^\infty(B) \) for each \( j \), then \( B[L] \) has admissible limits at almost all points of \( S \) and \( B[L]^* = u_1 \cdots u_N \) a.e. on \( S \).

In the proof below we use the notation \( \rho(z,w) = |\varphi_z(w)| \) for the pseudohyperbolic distance between \( z,w \in B \). We will use the well-known automorphism invariance of the pseudohyperbolic distance, i.e., the fact that \( \rho(z,w) = \rho(\varphi_a(z), \varphi_a(w)) \) for all \( a, z, w \in B \); see [34, Corollary 1.22].

**Proof.** Since every bounded pluriharmonic functions has admissible boundary values at almost all boundary points, (b) is an immediate consequence of (a).

We now show (a). So, assume that each \( u_j \) has an admissible limits at \( \zeta \in S \). Fix \( f \in A^2(B) \) and \( \alpha > 1 \). Easily modifying the inductive proof of Theorem 3.2(a), we only need to show

\[
\lim_{z \to \zeta, z \in \Gamma_\alpha(\zeta)} \int_B |u_1 \circ \varphi_z - u_1^*(\zeta)|^2 |f|^2 dV = 0.
\]

In order to prove this, we may assume \( u_1^*(\zeta) = 0 \) for simplicity. Let \( z \in \Gamma_\alpha(\zeta) \). Given \( 0 < \delta < 1 \), we have

\[
\|(u_1 \circ \varphi_z)f\|^2_{A^2(S)} \leq \int_{B \setminus \delta B} + \int_{\delta B} |u_1 \circ \varphi_z|^2 |f|^2 dV := I_1 + I_2.
\]

The first term is easily treated by

\[
I_1 \leq \|u_1\|^2 \int_{B \setminus \delta B} |f|^2 dV.
\]

We now consider the second term. Let \( w \in \delta B \) and put \( a = \varphi_z(w) \). Since the pseudohyperbolic distance is automorphism invariant, we have

\[
\delta > |w| = \rho(\varphi_z(w), \varphi_z(0)) = \rho(a, z)
\]

and therefore

\[
1 - \delta^2 < 1 - \rho^2(a, z) = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - a \cdot \overline{z}|^2} < \frac{4(1 - |a|^2)}{1 - |z|^2}.
\]

Since \( z \in \Gamma_\alpha(\zeta) \), this yields \( 1 - \delta^2 < 2\alpha(1 - |a|^2)/|1 - z \cdot \overline{\zeta}| \). Also, since \( \rho(a,z) < \delta \), we have \( |1 - a \cdot \overline{z}| < C_\delta(1 - |a|^2) \). Thus

\[
|1 - a \cdot \overline{z}|^{1/2} \leq |1 - a \cdot \overline{\zeta}|^{1/2} + |1 - z \cdot \overline{\zeta}|^{1/2} < C_\alpha,\delta(1 - |a|^2)^{1/2}.
\]

This shows that, for each \( z \in \Gamma_\alpha(\zeta) \), \( \varphi_z \) maps \( \delta B \) into some fixed admissible approach region with vertex \( \zeta \), say \( \Gamma_\beta(\zeta) \), depending only on \( \alpha \) and \( \delta \). Consequently, we have

\[
I_2 \leq \sup_{u \in \varphi_z(\delta B) \subset \Gamma_\beta(\zeta)} \|u_1(w)\|^2 \|f\|^2.
\]

Since \( u_1^*(\zeta) = 0 \) and \( \varphi_z \to \zeta \) uniformly on \( \delta B \) as \( z \to \zeta \), this yields \( I_2 \to 0 \) as \( z \in \Gamma_\alpha(\zeta) \to \zeta \) for each fixed \( \delta \). Note that the right side of (5.5) is independent of \( \delta \) and tends to 0 as \( \delta \to 1 \). Thus, taking the limit \( z \in \Gamma_\alpha(\zeta) \to \zeta \) with \( \delta \) fixed and then taking the limit \( \delta \to 1 \) in (5.4), we conclude (5.3), as desired. The proof is complete. \( \square \)

As a consequence we have the following Bergman space analogue of Theorem 3.6 with the same proof. This also fails in the one dimensional case as in the Hardy space case by the “same” example.

**Theorem 5.6** \((n \geq 2)\). Let \( u_1, \ldots, u_N \in ph^\infty(B) \cap C(B \cup W) \) for some nonempty relatively open set \( W \subset S \). If \( T_{u_1} \cdots T_{u_N} \) is compact on \( A^2(B) \), then \( u_j = 0 \) for some \( j \).
Our observation above in the setting of Bergman space suggests the following remarks.

(1) In connection with Theorem 5.6, a finite rank product theorem seems to be worth mentioning. Under the weaker ordinary harmonicity hypothesis but with the same boundary continuous extension hypothesis on symbols, it is known by the authors [9, Theorem 1.1] that if $T_u T_v$ has finite rank on $A^2(B)$, then either $u = 0$ or $v = 0$.

(2) The Bergman space analogue (but not exactly the same) of Corollary 4.3 over the disk is also known by the authors ([8, Theorem 1.1]). More recently, they ([10, Theorem 1.1]) have extended such a result to the polydisk. However, their proofs do not extend to the ball case and so the ball analogue still remains open. The Bergman space analogue (in the sense of [8] and [10]) of Theorem 4.2 also remains open.

6. REMARKS/QUESTIONS

Our results in Section 4 naturally suggest to study $M$-harmonic functions of the form

$$
\sum_{j=1}^{N} f_j g_j
$$

where $f_j$ and $g_j$ are holomorphic on $B$ for each $j$. Two papers by Ahern and Rudin [2] for $N = 1$ and by Zheng [33] for $N = 2$ are earlier studies in this direction. Surprisingly, unlike the one-variable case, the main result of [2] shows that there is a wide class of non-constant holomorphic functions $f$ and $g$ such that $f g$ is $M$-harmonic on balls of dimension higher than 2.

On the other hand, results in [2] and [33] (also see Lemma 4.5) indicate that if suitable regularity such as membership in $H^{2n}(B)$ is satisfied by functions $f_j$ and $g_j$, then the function in (6.1) with $N \leq 2$ must be pluriharmonic. Such a phenomenon is not surprising in view of the fact that “genuine” $M$-harmonic functions do not have much freedom with regard to regularity near boundary. For example, if an $M$-harmonic function $\psi$ on $B$ has $n$-th order smoothness across boundary, then it must be already pluriharmonic; see [1] or [18]. A more precise version is as follows: If an $M$-harmonic function $\psi$ on $B$ satisfies the integral growth rate

$$
\left\{ \int_S |\mathcal{R}^n \psi(r \zeta)|^2 \, d\sigma(\zeta) \right\}^{1/2} = o \left( \log \frac{1}{1-r} \right) \quad \text{as} \quad r \to 1,
$$

then $\psi$ is pluriharmonic; see [2].

Now, recall that if the functions $f_j$ and $g_j$ in (6.1) are coming from holomorphic parts of bounded pluriharmonic functions as in our results, then they all belong to $BMOA(B)$ which is contained in $\cap_{0<\rho<\infty} H^p(B)$. Thus we are tempted to ask the following question.

**Question 6.1** ($n \geq 2$). Let $\psi$ be an $M$-harmonic function as in (6.1) and assume $f_j, g_j \in BMOA(B)$ (or some $H^p(B)$) for each $j = 1, \ldots, N$. Is then $\psi$ pluriharmonic?

As mentioned above, the answer to the above question is known to be yes for $N \leq 2$ and due to Zheng [33]. One good thing if the answer would be positive is that we can exploit more explicit information provided by Lemma 4.7 as in the proof of Theorem 4.8.

A related question, which is trivial for $n = 1$, in conjunction with Theorem 4.2 is as follows.

**Question 6.2** ($n \geq 2$). Let $\psi \in L^\infty(B)$ be $M$-harmonic and $f \in BMOA(B)$ (or some $H^p(B)$) be nonconstant. Assume $f \psi$ is $M$-harmonic on $B$. Is then $\psi$ holomorphic?
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