Bessel Equation of Order One-Half

- The Bessel Equation of order one-half is
\[ x^2 y'' + xy' + \left( x^2 - \frac{1}{4} \right) y = 0 \]

- We assume solutions have the form
\[ y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0, \ x > 0 \]

- Substituting these into the differential equation, we obtain
\[
\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} \\
+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{r+n} = 0
\]
Recurrence Relation

• Using results of previous slide, we obtain

\[
\sum_{n=0}^{\infty} \left[ (r+n)(r+n-1) + (r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0
\]

or

\[
\left( r^2 - \frac{1}{4} \right) a_0 x^r + \left[ (r+1)^2 - \frac{1}{4} \right] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ \left[ (r+n)^2 - \frac{1}{4} \right] a_n + a_{n-2} \right\} x^{r+n} = 0
\]

• The roots of indicial equation are \( r_1 = 1/2, \ r_2 = -1/2, \) and note that they differ by a positive integer.

• The recurrence relation is

\[
a_n(r) = - \frac{a_{n-2}(r)}{(r+n)^2 - 1/4}, \quad n = 2, 3, \ldots
\]
First Solution: Coefficients

- Consider first the case $r_1 = 1/2$. From the previous slide,
  \[
  \left( r^2 - \frac{1}{4} \right) a_0 x^r + \left( (r+1)^2 - \frac{1}{4} \right) a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ \left( (r+n)^2 - \frac{1}{4} \right) a_n + a_{n-2} \right\} x^{r+n} = 0
  \]

- Since $r_1 = 1/2$, $a_1 = 0$, and hence from the recurrence relation, $a_1 = a_3 = a_5 = ... = 0$.

For the even coefficients, we have

\[
a_{2m} = -\frac{a_{2m-2}}{(1/2 + 2m)^2 - 1/4} = -\frac{a_{2m-2}}{2m(2m+1)}, \quad m = 1, 2, ...
\]

- It follows that
  \[a_2 = -\frac{a_0}{3!}, \quad a_4 = -\frac{a_2}{5\cdot 4} = \frac{a_0}{5!}, \ldots\]

and

\[
a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad m = 1, 2, ...
\]
It follows that the first solution of our equation is, for \( a_0 = 1 \),

\[
y_1(x) = x^{1/2} \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \right], \quad x > 0
\]

\[
= x^{-1/2} \left[ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \right], \quad x > 0
\]

\[
= x^{-1/2} \sin x, \quad x > 0
\]

The Bessel function of the first kind of order one-half, \( J_{1/2} \), is defined as

\[
J_{1/2}(x) = \left( \frac{2}{\pi} \right)^{1/2} y_1(x) = \left( \frac{2}{\pi x} \right)^{1/2} \sin x, \quad x > 0
\]
Second Solution: Even Coefficients

- Now consider the case $r_2 = -1/2$. We know that
  \[
  (r^2 - 1/4)a_0 x^r + \left[ (r + 1)^2 - \frac{1}{4} \right]a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ \left[ (r + n)^2 - \frac{1}{4} \right]a_n + a_{n-2} \right\} x^{r+n} = 0
  \]
- Since $r_2 = -1/2$, $a_1$ is arbitrary. For the even coefficients,
  \[
a_{2m} = -\frac{a_{2m-2}}{(-1/2 + 2m)^2 - 1/4} = -\frac{a_{2m-2}}{2m(2m-1)}, \quad m = 1, 2, \ldots
  \]
- It follows that
  \[
a_2 = -\frac{a_0}{2!}, \quad a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \ldots
  \]
  and
  \[
a_{2m} = \frac{(-1)^m a_0}{(2m)!}, \quad m = 1, 2, \ldots
  \]
Second Solution: Odd Coefficients

• For the odd coefficients,

\[ a_{2m+1} = - \frac{a_{2m-1}}{(\frac{-1}{2} + 2m + 1)^2 - 1/4} = - \frac{a_{2m-1}}{2m(2m+1)}, \quad m = 1, 2, \ldots \]

• It follows that

and

\[ a_3 = -\frac{a_1}{3!}, \quad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}, \ldots \]

\[ a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!}, \quad m = 1, 2, \ldots \]
Second Solution

- Therefore

\[ y_2(x) = x^{-1/2} \left[ a_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \right], \quad x > 0 \]

\[ = x^{-1/2} \left[ a_0 \cos x + a_1 \sin x \right], \quad x > 0 \]

- The second solution is usually taken to be the function

\[ J_{-1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \cos x, \quad x > 0 \]

where \( a_0 = (2/\pi)^{1/2} \) and \( a_1 = 0 \).

- The general solution of Bessel’s equation of order one-half is

\[ y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) \]
Graphs of Bessel Functions, Order One-Half

- Graphs of $J_{1/2}$, $J_{-1/2}$ are given below.
- Note behavior of $J_{1/2}$, $J_{-1/2}$ similar to $J_0$, $Y_0$ for large $x$, with phase shift of $\pi/4$.

\[
J_{-1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \cos x, \quad J_{1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \sin x
\]

\[
J_0(x) \approx \left( \frac{2}{\pi x} \right)^{1/2} \cos \left( x - \frac{x}{4} \right), \quad Y_0(x) \approx \left( \frac{2}{\pi x} \right)^{1/2} \sin \left( x - \frac{x}{4} \right), \text{ as } x \to \infty
\]
clear all; clc; clf; hold on

x = 0:0.1:10;

J1 = besselj(1/2,x);
plot(x,J1,'r-','LineWidth',2);

J2 = besselj(-1/2,x);
plot(x,J2,'b-','LineWidth',2);

xlabel('x','fontsize',30)
ylabel('y','fontsize',30,'rotation',0)
grid on;

legend('y = J_{1/2}(x)','y = J_{-1/2}(x)')
axis([0 10 -0.5 1.5])
box on
set(gca,'fontsize',30)
Bessel Equation of Order One

- The Bessel Equation of order one is
  \[ x^2 y'' + xy' + \left(x^2 - 1\right)y = 0 \]

- We assume solutions have the form
  \[ y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0, \ x > 0 \]

- Substituting these into the differential equation, we obtain
  \[
  \sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} \\
  + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0
  \]
Recurrence Relation

• Using the results of the previous slide, we obtain
  \[
  \sum_{n=0}^{\infty} [(r+n)(r+n-1) + (r+n)-1] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0
  \]

  or
  \[
  (r^2 - 1)a_0 x^r + [(r+1)^2 - 1] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ [(r+n)^2 - 1] a_n + a_{n-2} \right\} x^{r+n} = 0
  \]

• The roots of indicial equation are \( r_1 = 1, \ r_2 = -1 \), and note that they differ by a positive integer.

• The recurrence relation is
  \[
  a_n(r) = -\frac{a_{n-2}(r)}{(r+n)^2 - 1}, \ \ n = 2, 3, \ldots
  \]
First Solution: Coefficients

- Consider first the case $r_1 = 1$. From previous slide,
  \[
  (r^2 - 1)a_0 x^r + [(r+1)^2 - 1] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ [(r+n)^2 - 1] a_n + a_{n-2} \right\} x^{r+n} = 0
  \]

- Since $r_1 = 1$, $a_1 = 0$, and hence from the recurrence relation, $a_1 = a_3 = a_5 = \ldots = 0$. For the even coefficients, we have
  \[
a_{2m} = -\frac{a_{2m-2}}{(1+2m)^2 - 1} = -\frac{a_{2m-2}}{2^2(m+1)m}, \quad m = 1, 2, \ldots
  \]

- It follows that
  \[
a_2 = -\frac{a_0}{2^2 \cdot 2 \cdot 1}, \quad a_4 = -\frac{a_2}{2^2 \cdot 3 \cdot 2} = \frac{a_0}{2^4 \cdot 3!2!}, \ldots
  \]
  and
  \[
a_{2m} = \frac{(-1)^m a_0}{2^{2m} (m+1)! m!}, \quad m = 1, 2, \ldots
  \]
Bessel Function of First Kind, Order One

- It follows that the first solution of our differential equation is
  \[ y_1(x) = a_0 x \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m (m+1)!m!} x^{2m} \right], \quad x > 0 \]

- Taking \( a_0 = 1/2 \), the Bessel function of the first kind of order one, \( J_1 \), is defined as
  \[ J_1(x) = \frac{x}{2} \left[ \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m (m+1)!m!} x^{2m} \right], \quad x > 0 \]

- The series converges for all \( x \) and hence \( J_1 \) is analytic everywhere.
Second Solution

- For the case $r_1 = -1$, a solution of the form
  \[ y_2(x) = a J_1(x) \ln x + x^{-1} \left[ 1 + \sum_{n=1}^{\infty} c_n x^{2n} \right], \quad x > 0 \]
  is guaranteed by Theorem 5.7.1.
- The coefficients $c_n$ are determined by substituting $y_2$ into the ODE and obtaining a recurrence relation, etc. The result is:
  \[ y_2(x) = -J_1(x) \ln x + x^{-1} \left[ 1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^m m! (m-1)!} x^{2n} \right], \quad x > 0 \]
  where $H_k$ is as defined previously.
- Note that $J_1 \to 0$ as $x \to 0$ and is analytic at $x = 0$, while $y_2$ is unbounded at $x = 0$ in the same manner as $1/x$. 
The second solution, the Bessel function of the second kind of order one, is usually taken to be the function
\[ Y_1(x) = \frac{2}{\pi} \left[ -y_2(x) + (\gamma - \ln 2)J_1(x) \right], \quad x > 0 \]
where \( \gamma \) is the Euler-Mascheroni constant.

The general solution of Bessel’s equation of order one is
\[ y(x) = c_1 J_1(x) + c_2 Y_1(x), \quad x > 0 \]

Note that \( J_1, Y_1 \) have same behavior at \( x = 0 \) as observed on previous slide for \( J_1 \) and \( y_2 \).
clear all; clc; clf; hold on

x = 0:0.1:10;

J1 = besselj(1,x);
plot(x,J1,'r-','LineWidth',2);

Y1 = bessely(1,x);
plot(x,Y1,'b-','LineWidth',2);

xlabel('x','fontsize',30)
ylabel('y','fontsize',30,'rotation',0)
grid on;

legend('y = J_1(x)','y = Y_1(x)')
set(gca,'fontsize',30)
axis([0 10 -0.5 1])
box on