Nonhomogeneous equation

- Recall the nonhomogeneous equation
  \[ y'' + p(t)y' + q(t)y = g(t) \]
  where \( p, q, g \) are continuous functions on an open interval \( I \).

- The associated homogeneous equation is
  \[ y'' + p(t)y' + q(t)y = 0 \]
Variation of Parameters

- In this section we will learn the **variation of parameters method** to solve the nonhomogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to homogeneous equation.

- Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals need to be evaluated, and this can present difficulties.
Example

Find a particular solution of
\[ y'' + 9y = 2 \csc t \]

- We cannot use method of undetermined coefficients since \( g(t) \) is a quotient of \( \sin t \) or \( \cos t \), instead of a sum or product.

- Recall that the solution to the homogeneous equation is
  \[ y_C(t) = c_1 \cos 3t + c_2 \sin 3t \]

- To find a particular solution to the nonhomogeneous equation, we begin with the form
  \[ y(t) = u_1(t) \cos 3t + u_2(t) \sin 3t \]

- Then
  \[ y'(t) = u_1'(t) \cos 3t - 3u_1(t) \sin 3t + u_2'(t) \sin 3t + 3u_2(t) \cos 3t \]
  or \[ y'(t) = -3u_1(t) \sin 3t + 3u_2(t) \cos 3t + u_1'(t) \cos 3t + u_2'(t) \sin 3t \]
(1) Derivatives

- From the previous slide,
  \[ y'(t) = -3u_1(t)\sin 3t + 3u_2(t)\cos 3t + u_1'(t)\cos 3t + u_2'(t)\sin 3t \]

- Note that we need two equations to solve for \( u_1 \) and \( u_2 \). The first equation is the differential equation. To get a second equation, we will require
  \[ u_1'(t)\cos 3t + u_2'(t)\sin 3t = 0 \]

- Then
  \[ y'(t) = -3u_1(t)\sin 3t + 3u_2(t)\cos 3t \]

- Next,
  \[ y''(t) = -3u_1'(t)\sin 3t - 9u_1(t)\cos 3t + 3u_2'(t)\cos 3t - 9u_2(t)\sin 3t \]
(2) Two equations

- Recall that our differential equation is
  \[ y'' + 9y = 2 \csc t \]

- Substituting \( y'' \) and \( y \) into this equation, we obtain
  \[
  -3u'_1(t) \sin 3t - 9u_1(t) \cos 3t + 3u'_2(t) \cos 3t - 9u_2(t) \sin 3t \\
  + 9\left( u_1(t) \cos 3t + u_2(t) \sin 3t \right) = 2 \csc t
  \]

- This equation simplifies to
  \[-3u'_1(t) \sin 3t + 3u'_2(t) \cos 3t = 2 \csc t\]

- Thus, to solve for \( u_1 \) and \( u_2 \), we have the two equations:
  \[
  -3u'_1(t) \sin 3t + 3u'_2(t) \cos 3t = 2 \csc t \\
  u'_1(t) \cos 3t + u'_2(t) \sin 3t = 0
  \]
(3) Solve $u_1$

- To find $u_1$ and $u_2$, we need to solve the equations
  \[ -3u_1'(t)\sin 3t + 3u_2'(t)\cos 3t = 2\csc t \]
  \[ u_1'(t)\cos 3t + u_2'(t)\sin 3t = 0 \]

- From second equation,
  \[ u_2'(t) = -u_1'(t)\frac{\cos 3t}{\sin 3t} \]

- Substituting this into the first equation,
  \[ -3u_1'(t)\sin 3t + 3\left[-u_1'(t)\frac{\cos 3t}{\sin 3t}\right]\cos 3t = 3\csc t \]
  \[ -3u_1'(t)\sin^2(3t) - 3u_1'(t)\cos^2(3t) = 2\csc t \sin 3t \]
  \[ -3u_1'(t)\left[\sin^2(3t) + \cos^2(3t)\right] = 2\left[\frac{3\sin t - 4\sin^3 t}{\sin t}\right] \]
  \[ u_1'(t) = -\frac{2}{3}\left[3 - 4\sin^2 t\right] \]
(4) Solve $u_2$

- From the previous slide,
  \[ u'_1(t) = -\frac{2}{3} [3 - 4 \sin^2 t], \quad u'_2(t) = -u'_1(t) \frac{\cos 3t}{\sin 3t} \]

- Then
  \[ u'_2(t) = -\frac{2}{3} [3 - 4 \sin^2 t] \cdot \left[ \frac{\cos 3t}{\sin 3t} \right] = -\frac{2}{3} [3 - 4 \sin^2 t] \left[ \frac{4 \cos^3 t - 3 \cos t}{3 \sin t - 4 \sin^3 t} \right] \]
  \[ = -\frac{2}{3} \left[ \frac{4 \cos^3 t}{\sin t} - \frac{3 \cos t}{\sin t} \right] = -\frac{8}{3} \cos t \cdot \cos^2 t + 2 \cot t \]

- Thus
  \[ u_1(t) = \int u'_1(t)dt = \int -\frac{2}{3} [3 - 4 \sin^2 t]dt = -\frac{2}{3} t + \frac{2}{3} \sin 2t + c_1 \]
  \[ u_2(t) = \int u'_2(t)dt = \int \left( -\frac{8}{3} \cot t \cdot \cos^2 t + 2 \cot t \right)dt = -\frac{2}{3} \cos 2t - \frac{8}{3} \ln |\sin t| + c_2 \]
(5) General solution

- Recall our equation and homogeneous solution $y_C$:
  
  $y'' + 9y = 2 \csc t, \quad y_C(t) = c_1 \cos 3t + c_2 \sin 3t$

- Using the expressions for $u_1$ and $u_2$ on the previous slide, the general solution to the differential equation is

  $y(t) = u_1(t) \cos 3t + u_2(t) \sin 3t + y_C(t)$

  $$= \left( -\frac{2}{3} t + \frac{2}{3} \sin 2t \right) \cos 3t + \left( -\frac{2}{3} \cos 2t - \frac{8}{3} \ln | \sin t | \right) \sin 3t + y_C(t)$$

  $$= -\frac{2}{3} t \cos 3t + \frac{2}{3} \sin 2t \cos 3t - \frac{2}{3} \cos 2t \sin 3t - \frac{8}{3} \ln | \sin t | \sin 3t$$

  $$+ c_1 \cos 3t + c_2 \sin 3t.$$
Theorem

Consider the equations
\[ y'' + p(t)y' + q(t)y = g(t) \quad (1) \]
\[ y'' + p(t)y' + q(t)y = 0 \quad (2) \]

If the functions \( p, q, \) and \( g \) are continuous on an open interval \( I, \) and if \( y_1 \) and \( y_2 \) are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

\[ Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \, dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \, dt \]

and the general solution is

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \]
Suppose $y_1, y_2$ are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation above, where we note that the coefficient on $y''$ is 1.

To find $u_1$ and $u_2$, we need to solve the equations
\[ u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \]
\[ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \]

Doing so, and using the Wronskian, we obtain
\[ u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \]

Thus
\[ u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \, dt + c_1, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \, dt + c_2 \]