Nonhomogeneous Equations

• We consider the nonhomogeneous equation

\[ y'' + p(t)y' + q(t)y = g(t) \]

where \( p, q, g \) are continuous functions on an open interval \( I \).

• The associated homogeneous equation is

\[ y'' + p(t)y' + q(t)y = 0 \]

• In this section, to solve the nonhomogeneous equation, we will learn the method of undetermined coefficients, which relies on knowing solutions to homogeneous equation.
Theorem 1

• If $Y_1, Y_2$ are solutions of nonhomogeneous equation
  \[ y'' + p(t)y' + q(t)y = g(t) \]
  then $Y_1 - Y_2$ is a solution of the homogeneous equation
  \[ y'' + p(t)y' + q(t)y = 0 \]

• If $y_1, y_2$ form a fundamental solution set of
  homogeneous equation, then there exists constants $c_1, c_2$ such that
  \[ Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t) \]
Theorem 2

- The general solution of nonhomogeneous equation
  \[ y'' + p(t)y' + q(t)y = g(t) \]
  can be written in the form
  \[ y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \]
  where \( y_1, y_2 \) form a fundamental solution set of homogeneous equation, \( c_1, c_2 \) are arbitrary constants and \( Y \) is a particular solution to the nonhomogeneous equation.

\[ Y'' + p(t)Y' + q(t)Y = g(t) \]
Method of Undetermined Coefficients

- Recall the nonhomogeneous equation
  \[ y'' + p(t)y' + q(t)y = g(t) \]
  with general solution
  \[ y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \]

- In this section we use the method of undetermined coefficients to find a particular solution \( Y \) to the nonhomogeneous equation, assuming we can find solutions \( y_1, y_2 \) for the homogeneous case.

- The method of undetermined coefficients is usually limited to when \( p \) and \( q \) are constant, and \( g(t) \) is a polynomial, exponential, sine or cosine function.
Example 1: Exponential $g(t)$

Consider the nonhomogeneous equation

$$y'' - y' + 4y = 2e^{3t}$$

- We seek $Y$ satisfying this equation. Since exponentials replicate through differentiation, a good start for $Y$ is:
  $$Y(t) = Ae^{3t} \Rightarrow Y'(t) = 3Ae^{3t}, \quad Y''(t) = 9Ae^{3t}$$

- Substituting these derivatives into differential equation,
  $$9Ae^{3t} - 3Ae^{3t} + 4Ae^{3t} = 2e^{3t}$$
  $$\iff 10Ae^{3t} = 2e^{3t} \iff A = 1/5$$

- Thus a particular solution to the nonhomogeneous ODE is
  $$Y(t) = \frac{1}{5}e^{3t}.$$
Example 2: Sine $g(t)$

Consider the nonhomogeneous equation

$$y'' - y' + 4y = 3\sin t$$

- We seek $Y$ satisfying this equation. Since sines replicate through differentiation, a good start for $Y$ is:
  $$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t, \quad Y''(t) = -A\sin t$$

- Substituting these derivatives into differential equation,
  $$-A\sin t - A\cos t + 4A\sin t = 3\sin t$$
  $$\iff (3A - 3)\sin t - 3A\cos t = 0$$
  $$\iff c_1 \sin t + c_2 \cos t = 0$$

- Since $\sin(x)$ and $\cos(x)$ are linearly independent (they are not multiples of each other), we must have $c_1 = c_2 = 0$, and hence $3A - 3 = A = 0$, which is impossible.
Particular solution

- Our next attempt at finding a $Y$ is

$$Y(t) = A \sin t + B \cos t$$

$$\Rightarrow Y'(t) = A \cos t - B \sin t, \quad Y''(t) = -A \sin t - B \cos t$$

- Substituting these derivatives into ODE, we obtain

$$(-A \sin t - B \cos t) - (A \cos t - B \sin t) + 4(A \sin t + B \cos t) = 3 \sin t$$

$$\Leftrightarrow (3A + B) \sin t + (-A + 3B) \cos t = 3 \sin t$$

$$\Leftrightarrow 3A + B = 3, \quad -A + 3B = 0$$

$$\Leftrightarrow A = 9/10, \quad B = 3/10$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{9}{10} \sin t + \frac{3}{10} \cos t$$
Example 3: Polynomial g(t)

Consider the nonhomogeneous equation

\[ y'' - y' + 4y = 2t^2 - 1 \]

- We seek \( Y \) satisfying this equation. We begin with

\[ Y(t) = At^2 + Bt + C \Rightarrow Y'(t) = 2At + B, \quad Y''(t) = 2A \]

- Substituting these derivatives into differential equation,

\[ 2A - (2At + B) + 4(At^2 + Bt + C) = 2t^2 - 1 \]

\[ \iff 4At^2 - (2A + 4B)t + (2A - B + C) = 2t^2 - 1 \]

\[ \iff 4A = 2, \quad -2A + 4B = 0, \quad 2A - B + 4C = -1 \]

\[ \iff A = 1/2, \quad B = 1/4, \quad C = -7/16 \]

- Thus a particular solution to the nonhomogeneous ODE is

\[ Y(t) = \frac{1}{2}t^2 + \frac{1}{4}t - \frac{7}{16} \].
Example 4: Product g(t)

Consider the nonhomogeneous equation

\[ y'' - y' + 4y = -2e^t \cos 3t \]

- We seek \( Y \) satisfying this equation, as follows:

\[
Y(t) = Ae^t \cos 3t + Be^t \sin 3t \\
Y'(t) = Ae^t \cos 3t - 3Ae^t \sin 3t + Be^t \sin 3t + 3Be^t \cos 3t \\
= (A + 3B)e^t \cos 3t + (-3A + B)e^t \sin 3t \\
Y''(t) = (A + 3B)e^t \cos 3t - 3(A + 3B)e^t \sin 3t + (-3A + B)e^t \sin 3t \\
+ 3(-3A + B)e^t \cos 3t \\
= (-8A + 6B)e^t \cos 3t + (-6A - 8B)e^t \sin 3t
\]

- Substituting derivatives into ODE and solving for \( A \) and \( B \):

\[
A = \frac{5}{6}, \quad B = \frac{1}{6} \quad \Rightarrow \quad Y(t) = \frac{5}{6} e^t \cos 3t + \frac{1}{6} e^t \sin 3t
\]
Theorem 3 : Sum g(t)

- Consider again our general nonhomogeneous equation
  \[ y'' + p(t)y' + q(t)y = g(t) \]

- Suppose that \( g(t) \) is sum of functions:
  \[ g(t) = g_1(t) + g_2(t) \]

- If \( Y_1, Y_2 \) are solutions of
  \[ y'' + p(t)y' + q(t)y = g_1(t) \]
  \[ y'' + p(t)y' + q(t)y = g_2(t) \]
  respectively, then \( Y_1 + Y_2 \) is a solution of the nonhomogeneous equation above.
Example 5: Sum g(t)

Consider the equation

\[ y'' - y' + 4y = 2e^{3t} + 3 \sin t - 2e^t \cos 3t \]

- Our equations to solve individually are
  \[ y'' - y' + 4y = 2e^{3t} \]
  \[ y'' - y' + 4y = 3 \sin t \]
  \[ y'' - y' + 4y = -2e^t \cos 3t \]

- Our particular solution is then

\[ Y(t) = \frac{1}{5} e^{2t} + \frac{9}{10} \sin t + \frac{3}{10} \cos t + \frac{5}{6} e^t \cos 3t + \frac{1}{6} e^t \sin 3t. \]
clear all; clc; clf; hold on

t = 0:0.05:1.5; % define grid of values in t-direction

y = 1/5*exp(2*t) + 9/10*sin(t) + 3/10*cos(t) ...
+ 5/6*exp(t).*cos(3*t) + 1/6*exp(t).*sin(3*t); % particular solution
plot(t,y,'b-','LineWidth',2); % plot solution y(t)

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
grid on;

axis([0 1.5 0 3.5])
Example 6

Consider the equation

\[ y'' + 9y = 4 \cos 3t \]

- We seek \( Y \) satisfying this equation. We begin with

\[ Y(t) = A \sin 3t + B \cos 3t \]

\[ \Rightarrow Y'(t) = 3A \cos 3t - 3B \sin 3t, \quad Y''(t) = -9A \sin 3t - 9B \cos 3t \]

- Substituting these derivatives into ODE:

\[
\begin{align*}
(-9A \sin 3t - 9B \cos 3t) + 9(A \sin 3t + B \cos 3t) &= 4 \cos 3t \\
(-9A + 9A) \sin 3t + (-9B + 9B) \cos 3t &= 4 \cos 3t \\
0 &= 4 \cos 3t
\end{align*}
\]

- Thus no particular solution exists of the form

\[ Y(t) = A \sin 3t + B \cos 3t \]
**Particular solution**

- Thus no particular solution exists of the form
  \[ Y(t) = A \sin 3t + B \cos 3t \]

- To help understand why, recall that we found the corresponding homogeneous solution in Section 3.4 notes:
  \[ y'' + 9y = 0 \quad \Rightarrow \quad y(t) = c_1 \cos 3t + c_2 \sin 3t \]

- Thus our assumed particular solution solves homogeneous equation
  \[ y'' + 9y = 0 \]

  instead of the nonhomogeneous equation.
  \[ y'' + 9y = 4 \cos 3t \]
Our next attempt at finding a \( Y \) is:

\[
Y(t) = At \sin 3t + Bt \cos 3t
\]

\[
Y'(t) = A \sin 3t + 3A \cos 3t + B \cos 3t - 3Bt \sin 3t
\]

\[
Y''(t) = 3A \cos 3t + 3A \cos 3t - 9A \sin 3t - 3B \sin 3t - 3B \sin 3t - 9Bt \cos 3t
\]

\[
= 6A \cos 3t - 6B \sin 3t - 9A \sin 3t - 9Bt \cos 3t
\]

Substituting derivatives into ODE,

\[
6A \cos 3t - 6B \sin 3t = 4 \cos 3t
\]

\[\Rightarrow A = \frac{2}{3}, \quad B = 0\]

\[\Rightarrow Y(t) = \frac{2}{3} t \sin 3t.\]
clear all; clc; clf; hold on

t = 0:0.05:10; % define grid of values in t-direction

y = 3/4*t.*sin(2*t); % particular solution y(t)
plot(t,y,'b-','LineWidth',2); % plot solution y(t)

xlabel('t','fontSize',30)
ylabel('y(t)','fontSize',30,'rotation',0)
grid on;
box on

axis([0 10 -8 8])