Homogeneous Equations, Initial Values

• Consider homogeneous 2nd order equations:
  \[ ay'' + by' + cy = 0 \]

• Initial conditions are:
  \[ y(t_0) = y_0, \quad y'(t_0) = y'_0 \]

• Solution passes through \((t_0, y_0)\), and slope of solution at \((t_0, y_0)\) is \(y'_0\).
Characteristic Equation

- To solve the 2\textsuperscript{nd} order equation with constant coefficients, 
  \[ ay'' + by' + cy = 0, \]

(1) Assume a solution of the form \( y = e^{rt} \).

Substituting into the differential equation, we obtain

\[ ar^2 e^{rt} + be^{rt} + ce^{rt} = 0 \]

Since \( e^{rt} \neq 0 \), we have \( ar^2 + br + c = 0 \).

(2) We have the \textbf{characteristic equation} of the differential equation: \( ar^2 + br + c = 0 \).

(3) Solve \( r \) by factoring or using quadratic formula.
General Solution

\[ ar^2 + br + c = 0, \]
we obtain two solutions, \( r_1 \) and \( r_2 \).

\[ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

- Then:
  - The roots \( r_1, r_2 \) are real & \( r_1 \neq r_2 \).
  - The roots \( r_1, r_2 \) are real & \( r_1 = r_2 \).
  - The roots \( r_1, r_2 \) are complex.
- We assume \( r_1, r_2 \) are real, and \( r_1 \neq r_2 \).
- The **general solution** has the form

\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \]
Initial Conditions

\[ ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0, \]

We find \( c_1 \) and \( c_2 \) using the general form & initial:

\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \]

\[
\begin{align*}
    c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\
    c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y'_0
\end{align*}
\]

\[ \implies \quad c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \quad c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} e^{-r_2 t_0} \]

- As \( r_1 \neq r_2 \), a solution with form \( y = e^{rt} \) exists, for any initial conditions.
Example

Consider the initial value problem

\[ 3y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0 \]

(1) Assume exponential: \( y(t) = e^{rt} \)

(2) We have characteristic equation

\[ 3r^2 - r - 2 = 0 \iff (3r + 2)(r - 1) = 0 \]

(3) two solutions, \( r_1 = -2/3 \) and \( r_2 = 1 \).

Thus \[ y(t) = c_1 e^{\frac{-2}{3}t} + c_2 e^t \]
(4) Use the initial condition:

\[
\begin{align*}
    c_1 + c_2 &= 1 \\
    -\frac{2}{3}c_1 + c_2 &= 0
\end{align*}
\]

\[\Rightarrow c_1 = \frac{3}{5}, \quad c_2 = \frac{2}{5}\]

• Thus

\[y(t) = \frac{3}{5}e^{\frac{2}{3}t} + \frac{2}{5}e^t.\]
clear all; clc; clf; hold on

t = 0:0.1:3;
y = 3/5*exp(-2/3*t) + 2/5*exp(t);

plot(t,y,'b-','LineWidth',2);

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
grid on;

set(gca,'fontsize',30)
axis([0 3 0 10])
box on