• The equations of the form $y' = f(y)$ are called autonomous equations.

For example,

- $y' = ay + by^2$, $y' = e^y - 1$

- The logistic equation

$$\frac{dy}{dt} = r \left( 1 - \frac{y}{K} \right) y$$
Equilibrium solutions

- **Equilibrium solutions** of a general first order autonomous equation \( y' = f(y) \) can be found by locating roots of \( f(y) = 0 \).
- These roots of \( f(y) \) are called **critical points**.
- For example, the critical points of the logistic equation
  \[
  \frac{dy}{dt} = r \left(1 - \frac{y}{K}\right)y
  \]
  are \( y = 0 \) and \( y = K \).

Thus, critical points are constant functions (equilibrium solutions) in this setting.
Logistic Equation

- The logistic equation is
  \[
  \frac{dy}{dt} = r \left(1 - \frac{y}{K}\right)y
  \]

  where \( K = r/a \), \( r \) is the intrinsic growth rate, and \( K \) is the carrying capacity.

- A direction field for the logistic equation with \( r = 1 \) and \( K = 10 \) is given here.
Logistic Equation with Equilibrium Solutions

- Our logistic equation is
  \[ \frac{dy}{dt} = r \left( 1 - \frac{y}{K} \right) y, \quad r, K > 0 \]

- Two equilibrium solutions are \( y = \phi_1(t) = 0, \quad y = \phi_2(t) = K. \)

In direction field below, with \( r = 1, \ K = 10, \) note behavior of solutions near equilibrium solutions:
  \( y = 0 \) is unstable,
  \( y = 10 \) is asymptotically stable.
clear all; clc; clf;
hold on; grid on
r = 1; K = 10;
t = 0:0.5:15; % define grid of values in t-direction
y = -5:0.5:15; % define grid of values in y-direction

[T,Y] = meshgrid(t, y); % creates 2D matrices
dT = ones(size(T)); % dt = 1 for all points
dY = r*(1-Y/K).*Y; % dy = r(1-y/K)y*dt; this is the ODE

N = sqrt(dT.^2 + dY.^2); % magnitude of arrows
dT = dT./N; dY = dY./N; % normalize arrows to get all same length
quiver(T,Y,dT,dY); % draw arrows (t,y) --> (t+dt,y+dy)

y1 = 0; y2 = K; % equilibrium solutions, y1 and y2
plot(t,y1+0*t,'r-','LineWidth',2);
plot(t,y2+0*t,'r-','LineWidth',2);
The Solutions of Logistic Equation

- Provided \( y \neq 0 \) and \( y \neq K \), we can rewrite the logistic ODE:

\[
\frac{dy}{(1 - y/K)y} = rdt
\]

- Expanding the left side using partial fractions,

\[
\frac{1}{(1 - y/K)y} = \frac{A}{1 - y/K} + \frac{B}{y} \implies 1 = Ay + B(1 - y/K) \implies B = 1, \quad A = y/K
\]

- Thus the logistic equation can be rewritten as

\[
\left(\frac{1 + 1/K}{y + 1 - y/K}\right)dy = rdt
\]

- Integrating the above result, we obtain

\[
\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C
\]
• We have
\[ \ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C \]

• If \( 0 < y_0 < K \), then \( 0 < y < K \)
and hence
\[ \ln y - \ln\left(1 - \frac{y}{K}\right) = rt + C \]

• Using properties of logs, we rewrite
\[ \ln\left(\frac{y}{1 - y/K}\right) = rt + C \iff \frac{y}{1 - y/K} = e^{rt+C} \iff \frac{y}{1 - y/K} = ce^{rt} \]

or \[ y = \frac{y_0K}{y_0 + (K-y_0)e^{-rt}}, \quad \text{where} \ y_0 = y(0) \]
• We have
\[ y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \quad \text{for } 0 < y_0 < K \]

• It can be shown that solution is also valid for \( y_0 > K \). Also, this solution contains equilibrium solutions \( y = 0 \) and \( y = K \).

• Hence solution to logistic equation is
\[ y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \]
Let \( y \) be biomass (in kg) of halibut population at time \( t \), with \( r = 0.71/\text{year} \) and \( K = 80.5 \times 10^6 \text{ kg} \). If \( y_0 = 0.20K \), find

(a) biomass 4 years later
(b) the time \( \tau \) such that \( y(\tau) = 0.60K \).

**Figure source**: [www.suite101.com/view_image_articles.cfm/1403384](http://www.suite101.com/view_image_articles.cfm/1403384)
Solution)
(a) For convenience, scale equation

\[
\frac{y}{K} = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rt}}
\]

Then

\[
\frac{y(4)}{K} = \frac{0.2}{0.2 + 0.8e^{-(0.71)(4)}} \approx 0.8106
\]

and hence

\[y(4) \approx 0.8106K \approx 65.25 \times 10^6 \text{ kg}\]
(b) Find time $\tau$ for which $y(\tau) = 0.60K$.

\[
\frac{y}{K} = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-\tau}}
\]

\[
0.60 = \frac{\frac{y_0}{K}}{\frac{y_0}{K} + \left(1 - \frac{y_0}{K}\right)e^{-\tau}}
\]

\[
0.60 \left[\frac{\frac{y_0}{K}}{\frac{y_0}{K} + \left(1 - \frac{y_0}{K}\right)e^{-\tau}}\right] = \frac{y_0}{K}
\]

\[
0.60 \frac{y_0}{K} + 0.60\left(1 - \frac{y_0}{K}\right)e^{-\tau} = \frac{y_0}{K}
\]

\[
e^{-\tau} = \frac{0.40\frac{y_0}{K}}{0.60\left(1 - \frac{y_0}{K}\right)} = \frac{2\frac{y_0}{K}}{3\left(1 - \frac{y_0}{K}\right)}
\]

\[
\tau = \frac{-1}{0.71 \ln\left(\frac{0.40}{3(0.80)}\right)} \approx 2.5236 \text{ years}
\]
clear all; clc; clf;
hold on
r = 0.71; K = 80.5*10^6;
K = K/K; r = 0.71/K;  % redefine scaling factors
t = 0:0.2:8; % define grid of values in t-direction
y = 0:0.05:3;  % define grid of values in y-direction

[T,Y] = meshgrid(t,y);  % creates 2D matrices
dT = ones(size(T));     % dt = 1 for all points
dY = r*(1-Y/K).*Y;      % dy = r(1-y/K)y*dt; this is the ODE
N = sqrt(dT.^2 + dY.^2);  % magnitude of arrows
dT = dT/N; dY = dY/N; % normalize arrows to get all same length
quiver(T,Y,dT,dY,'r'); % draw arrows (t,y) --> (t+dt,y+dy)

y0 = 0.2*K;
yy = y0*K./(y0+t^0+(K-y0)*exp(-r*t));
plot(t,yy+0^t,'b-','LineWidth',2);
axis([0 8 0 1.5])